SPACES OF GEOMETRICALLY GENERIC CONFIGURATIONS

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Abstract. Let X denote either $\mathbb{CP}^m$ or $\mathbb{C}^m$. We study certain analytic properties of the space $\mathcal{E}^n(X,gp)$ of ordered geometrically generic n-point configurations in X. This space consists of all $q=(q_1,...,q_n)\in X^n$ such that no $m+1$ of the points $q_1,...,q_n$ belong to a hyperplane in X. In particular, we show that for n big enough any holomorphic map $f\colon \mathcal{E}^n(\mathbb{CP}^m,gp)\to\mathcal{E}^n(\mathbb{CP}^m,gp)$ commuting with the natural action of the symmetric group $\mathbf{S}(n)$ in $\mathcal{E}^n(\mathbb{CP}^m,gp)$ is of the form $f(q)=\tau(q)q=(\tau(q)q_1,...,\tau(q)q_n)$, $q\in\mathcal{E}^n(\mathbb{CP}^m,gp)$, where $\tau\colon\mathcal{E}^n(\mathbb{CP}^m,gp)\to\mathbf{PSL}(m+1,\mathbb{C})$ is an $\mathbf{S}(n)$-invariant holomorphic map. A similar result holds true for mappings of the configuration space $\mathcal{E}^n(\mathbb{C}^m,gp)$.

Contents

1. Introduction

In this paper we study certain analytic properties of the spaces of geometrically generic point configurations in projective and affine spaces.

The most traditional non-ordered configuration space $\mathcal{C}^n=\mathcal{C}^n(X)$ of a complex space X consists of all n point subsets (*"configurations"*) $Q=\{q_1,...,q_n\}\subset X$.

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If X carries an additional geometric structure, it may be taken into account. Say if X is either the projective space $\mathbb{CP}^m$ or the affine space $\mathbb{C}^m$ and $n > m$ then the space $\mathcal{C}^n = \mathcal{C}^n(X, gp)$ of *geometrically generic configurations* consists of all n point subsets $Q \subset X$ such that no hyperplane in X contains more than m points of Q. The corresponding *ordered* configuration space $\mathcal{E}^n = \mathcal{E}^n(X, gp)$ consists of all $q = (q_1, ..., q_n) \in X^n$ such that the set $Q = \{q_1, ..., q_n\} \subset X$ belongs to $\mathcal{C}^n(X, gp)$. The left action of the symmetric group $\mathbf{S}(n)$ on $\mathcal{E}^n$ defined by $\sigma(q_1, ..., q_n) = (q_{\sigma^{-1}(1)}, ..., q_{\sigma^{-1}(n)})$ is free; the corresponding orbit map can be identified with the tautological covering map $p\colon \mathcal{E}^n \ni (q_1, ..., q_n) \mapsto \{q_1, ..., q_n\} \in \mathcal{C}^n$, which is an unbranched Galois covering with the Galois group $\mathbf{S}(n)$.

When $m = 1$, that is, $X = \mathbb{CP}^1$ or $X = \mathbb{C}^1$, the spaces of geometrically generic configurations in X coincide with the usual configuration spaces $\mathcal{C}^n(X)$ and $\mathcal{E}^n(X)$. But for $m > 1$ the spaces $\mathcal{C}^n(X, gp)$ and $\mathcal{E}^n(X, gp)$ seem even more natural since they both are Stein manifolds, unlike $\mathcal{C}^n(X)$ and $\mathcal{E}^n(X)$. In the same time, the spaces $\mathcal{C}^n(X, gp)$ and $\mathcal{E}^n(X, gp)$ have more complicated topology; for instance, their fundamental groups are highly nontrivial, whereas $\pi_1(\mathcal{C}^n(X)) = \mathbf{S}(n)$ and $\mathcal{E}^n(X)$ is simply connected.

Let $\operatorname{Aut}_g X$ be the subgroup of the holomorphic automorphism group $\operatorname{Aut} X$ of X consisting of all elements $A \in \operatorname{Aut} X$ that respect the geometrical structure of X. I. e., for $X = \mathbb{CP}^m$ we have $\operatorname{Aut}_g X = \mathbf{PSL}(m+1, \mathbb{C}) = \operatorname{Aut} X$, whereas for $X = \mathbb{C}^m$ the group $\operatorname{Aut}_g X = \mathbf{Aff}(m, \mathbb{C})$ of all affine transformations of $\mathbb{C}^m$ is much smaller than the huge group $\operatorname{Aut}(\mathbb{C}^m)$ of all biholomolomorphic (or polynomial) automorphisms of $\mathbb{C}^m$. Notice that in both cases $\operatorname{Aut}_g X$ is a complex Lie group. The natural action of $\operatorname{Aut}_g X$ in X induces the diagonal $\operatorname{Aut}_g X$ actions in $\mathcal{C}^n$ and $\mathcal{E}^n$ defined by

$$AQ = A\{q_1, ..., q_n\} \stackrel{\text{def}}{=\!=} \{Aq_1, ..., Aq_n\} \ \ \forall\, Q = \{q_1, ..., q_n\} \in \mathcal{C}^n$$

and

$$Aq = A(q_1, ..., q_n) \stackrel{\text{def}}{=\!=} (Aq_1, ..., Aq_n) \ \ \forall\, q = (q_1, ..., q_n) \in \mathcal{E}^n\,.$$

Therefore, any automorphism $T \in \operatorname{Aut}_g X$ gives rise to the holomorphic endomorphism (i.e., holomorphic self-mapping) F_T of $\mathcal{C}^n$ defined by $F_T(Q) = TQ = \{Tq_1, ..., Tq_n\}$. This example may be generalized by making T depending analytically on a configuration $Z \in \mathcal{C}^n$; a rigorous definition looks as follows.

Definition 1.1. A holomorphic endomorphism F of $\mathcal{C}^n$ is said to be *tame* if there is a holomorphic map $T\colon \mathcal{C}^n \to \operatorname{Aut}_g X$ such that $F(Q) = F_T(Q) \stackrel{\text{def}}{=\!=} T(Q)Q$ for all $Q \in \mathcal{C}^n$. Similarly, a holomorphic endomorphism f of $\mathcal{E}^n$ is called *tame* if there are an $\mathbf{S}(n)$-invariant holomorphic map $\tau\colon \mathcal{E}^n \to \operatorname{Aut}_g X$ and a permutation $\sigma \in \mathbf{S}(n)$ such that $f(q) = f_{\tau,\sigma}(q) \stackrel{\text{def}}{=\!=} \sigma\tau(q)q$ for all $q \in \mathcal{E}^n$.

When $X = \mathbb{C}^m$, a holomorphic endomorphism F of $\mathcal{C}^n(\mathbb{C}^m, gp)$ is said to be *quasitame* if there is a holomorphic map $T\colon \mathcal{C}^n(\mathbb{C}^m, gp) \to \mathbf{PSL}(m+1, \mathbb{C})$ such that $F(Q) = F_T(Q) \stackrel{\text{def}}{=\!=} T(Q)Q$ for all $Q \in \mathcal{C}^n(\mathbb{C}^m, gp)$.[1] Similarly, a holomorphic endomorphism f of $\mathcal{E}^n = \mathcal{E}^n(\mathbb{C}^m, gp)$ is *quasitame* if there are an $\mathbf{S}(n)$-invariant holomorphic map $\tau\colon \mathcal{E}^n(\mathbb{C}^m, gp) \to \mathbf{PSL}(m+1, \mathbb{C})$ and a permutation $\sigma \in \mathbf{S}(n)$ such that $\tau(q)q_1, ..., \tau(q)q_n \in \mathbb{C}^m$ and $f(q) = \sigma\tau(q)q$ for any $q = (q_1, ..., q_n) \in \mathcal{E}^n(\mathbb{C}^m, gp)$. ◯

[1]Notice that the latter condition implies that $T(Q)Q \subset \mathbb{C}^m$ for any $Q \in \mathcal{C}^n(\mathbb{C}^m, gp)$.

The left $\mathbf{S}(n)$ action on $\mathcal{E}^n(X,gp)$ induces the left $\mathbf{S}(n)$ action on the set of all mappings $f\colon \mathcal{E}^n(X,gp)\to\mathcal{E}^n(X,gp)$ defined by

$$\sigma f=\sigma(f_1,...,f_n)=(f_{\sigma^{-1}(1)},...,f_{\sigma^{-1}(n)})$$

for $f=(f_1,...,f_n)\colon \mathcal{E}^n(X,gp)\to\mathcal{E}^n(X,gp)$ and $\sigma\in\mathbf{S}(n)$.

Definition 1.2. A continuous map $f\colon \mathcal{E}^n(X,gp)\to\mathcal{E}^n(X,gp)$ is said to be *strictly equivariant* if there exists an automorphism α of the group $\mathbf{S}(n)$ such that

$$f(\sigma q)=\alpha(\sigma)f(q)\ \text{ for all } q\in\mathcal{E}^n(X) \text{ and } \sigma\in\mathbf{S}(n)\,.$$

○

Remark 1.3. Any tame endomorphism $f=f_{\tau,\sigma}$ of $\mathcal{E}^n$ is strictly equivariant; the corresponding automorphism $\alpha\in\operatorname{Aut}\mathbf{S}(n)$ is just the inner automorphism $s\mapsto\sigma s\sigma^{-1}$. The same holds true for quasitame maps. ○

The following theorem contains the main results of this work.

Theorem 1.4. *Let $m>1$, $n\geq m+3$ and $n\neq 2m+2$.*

a) Any strictly equivariant holomorphic map $f\colon \mathcal{E}^n(\mathbb{CP}^m,gp)\to\mathcal{E}^n(\mathbb{CP}^m,gp)$ is tame.

b) Any strictly equivariant holomorphic map $f\colon \mathcal{E}^n(\mathbb{C}^m,gp)\to\mathcal{E}^n(\mathbb{C}^m,gp)$ is quasitame.

In more detail, for $m>1$, $n\geq m+3$, $n\neq 2m+2$ and a strictly equivariant holomorphic map $f\colon \mathcal{E}^n(X,gp)\to\mathcal{E}^n(X,gp)$, there exist an $\mathbf{S}(n)$-invariant holomorphic map $\tau\colon \mathcal{E}^n(X,gp)\to\mathbf{PSL}(m+1,\mathbb{C})$ and a permutation $\sigma\in\mathbf{S}(n)$ such that

$$f(q)=\sigma\tau(q)q=(\tau(q)q_{\sigma^{-1}(1)},...,\tau(q)q_{\sigma^{-1}(n)}) \tag{1.1}$$

for all $q=(q_1,\ldots,q_n)\in\mathcal{E}^n(X,gp)$.

Remark 1.5. For a holomorphic endomorphism f of $\mathcal{E}^n(X,gp)$ commuting with the $\mathbf{S}(n)$-action the representation (1.1) can be simplified as follows: there is an $\mathbf{S}(n)$-invariant holomorphic map $\tau\colon \mathcal{E}^n(X,gp)\to\mathbf{PSL}(m+1,\mathbb{C})$ such that

$$f(q)=\tau(q)q\ \text{ for all } q\in\mathcal{E}^n(X,gp)\,.$$

○

Remark 1.6. The holomorphic endomorphisms of the spaces $\mathcal{C}^n(\mathbb{CP}^1)$ and $\mathcal{C}^n(\mathbb{C})$ were completely described by V. Lin (see, for instance, [6], [8] and [11]). Moreover, due to the works of V. Zinde [17]-[21] and the author [3], such a description is known now for holomorphic endomorphisms of traditional configuration spaces of all non-hyperbolic algebraic curves Γ. In all quoted papers, the first important step includes a purely algebraic investigation of the fundamental groups $\pi_1(\mathcal{C}^n(\Gamma))$ and $\pi_1(\mathcal{E}^n(\Gamma))$, which are the braid group and the pure braid group of Γ, respectively. Namely, it is shown that for $n>4$ any endomorphism of $\pi_1(\mathcal{C}^n(\Gamma))$ with a non-abelian image preserves the subgroup $\pi_1(\mathcal{E}^n(\Gamma))$. This property ensures the lifting of any "sufficiently non-trivial" holomorphic self-map F of $\mathcal{C}^n(\Gamma)$ to a strictly equivariant holomorphic self-map f of $\mathcal{E}^n(\Gamma)$, which, in turn, can be studied via certain analytic and combinatorial methods.

My original target in the present work was to use a similar machinery in order to study the holomorphic endomorphisms of the spaces $\mathcal{C}^n(\mathbb{CP}^m,gp)$ and $\mathcal{C}^n(\mathbb{C}^m,gp)$

for $m > 1$. I did not succeed however in investigating of the corresponding fundamental groups and therefore restricted myself to the study of strictly equivariant holomorphic endomorphisms of the corresponding ordered configuration spaces. ◯

REMARK 1.7. Theorem 1.4(b) is not complete, since at the moment I do not know whether there are strictly equivariant holomorphic endomorphisms of $\mathcal{E}^n(\mathbb{C}^m, gp)$ that are quasitame but not tame.

Although I think that Theorem 1.4 holds true for $n = 2m + 2$, in this case I could not overcome some technical difficulties which arise in the proof. ◯

The plan of the proof is as follows. Let $X = \mathbb{C}^m$ or $\mathbb{C}P^m$. To study a strictly equivariant holomorphic self-map f of the space $\mathcal{E}^n = \mathcal{E}^n(X, gp)$, we start with an explicit description of all non-constant holomorphic functions $\lambda\colon \mathcal{E}^n \to \mathbb{C} \setminus \{0, 1\}$.[2] The set L of all such maps is finite and separates points of a certain rather big submanifold $M \subset \mathcal{E}^n$ of complex codimension $m(m + 1)$. An endomorphism f induces a self-map f^* of L via

$$f^*\colon L \ni \lambda \mapsto f^*\lambda = \lambda \circ f \in L\,.$$

The map f^* carries important information about f. In order to investigate behaviour of f^* and then recover f, we endow L with the following simplicial structure. A subset $\Delta^s = \{\lambda_0, ..., \lambda_s\} \subseteq L$ is said to be an s-*simplex* whenever $\lambda_i/\lambda_j \in L$ for all distinct i, j. The action of $\mathbf{S}(n)$ in $\mathcal{E}^n$ induces a simplicial $\mathbf{S}(n)$ action in the complex L. The orbits of this action may be exhibited explicitly. On the other hand, the map $f^*\colon L \to L$ defined above is simplicial and preserves dimension of simplices. Since f is equivariant, f^* is nicely related to the $\mathbf{S}(n)$ action on L.

Studying all these things together, we find a holomorphic map

$$\tau\colon \mathcal{E}^n(X, gp) \to \mathbf{PSL}(m + 1, \mathbb{C})$$

and a permutation $\sigma \in \mathbf{S}(n)$ such that $f(q) = \sigma\tau(q)q$.

REMARK 1.8. The topology of the considered spaces is of great interest by itself. A. I. Barvinok calculated the first homology group of the ordered space $\mathcal{E}^n(\mathbb{C}^2, gp)$, see [1]. V. Moulton, [13], found the generators and some generating relations of the fundamental groups $\pi_1(\mathcal{E}^n(\mathbb{C}^m, gp))$ and $\pi_1(\mathcal{E}^n(\mathbb{CP}^m, gp))$. T. Terasoma, [15], proved that this set of generating relations is complete for the case $m = 2$. ◯

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2. Some properties of spaces of generic configurations

2.1. **Notation and definitions.** The spaces $\mathcal{E}^n$ of ordered geometrically generic configurations in $\mathbb{CP}^m$ or $\mathbb{C}^m$ have the following explicit algebraic description.

Any point $q \in (\mathbb{CP}^m)^n$ may be represented as a 'matrix'

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} [z_{1,1} : \cdots : z_{1,m+1}] \\ \vdots \\ [z_{n,1} : \cdots : z_{n,m+1}] \end{pmatrix} \in (\mathbb{CP}^m)^n\,, \tag{2.1}$$

[2]Compare with [11], [18], [19] and [3].

where $q_j = [z_{j,1} : \cdots : z_{j,m+1}] \in \mathbb{CP}^m$, $j = 1, ..., n$. For $m+1$ distinct indices $i_1, ..., i_{m+1} \in \{1, ..., n\}$, the determinant

$$d_{i_1,...,i_{m+1}}(q) = \begin{vmatrix} z_{i_1,1} & \dots & z_{i_1,m} & z_{i_1,m+1} \\ \vdots & \vdots & \vdots & \vdots \\ z_{i_{m+1},1} & \dots & z_{i_{m+1},m} & z_{i_{m+1},m+1} \end{vmatrix} \tag{2.2}$$

is a homogeneous polynomial of degree $m+1$ in the homogeneous coordinates $[z_{1,1} : \cdots : z_{1,m+1}]$, ...,$[z_{n,1} : \cdots : z_{n,m+1}]$. The space $\mathcal{E}^n(\mathbb{CP}^m, gp)$ consists of all matrices q of the form (2.1) such that $d_{i_1,...,i_{m+1}}(q) \neq 0$ for all distinct $i_1, ..., i_{m+1} \in \{1, ..., n\}$.

Similarly, the space $\mathcal{E}^n(\mathbb{C}^m, gp)$ consists of all matrices

$$q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} z_{1,1}, ..., z_{1,m}, 1 \\ \vdots \\ z_{n,1}, ..., z_{n,m}, 1 \end{pmatrix} \in (\mathbb{C}^m)^n \tag{2.3}$$

with all $q_j = (z_{j,1}, ..., z_{j,m}) \in \mathbb{C}^m$ such that

$$d_{i_1,...,i_{m+1}}(q) = \begin{vmatrix} z_{i_1,1} & \dots & z_{i_1,m} & 1 \\ \vdots & \vdots & \vdots & \vdots \\ z_{i_{m+1},1} & \dots & z_{i_{m+1},m} & 1 \end{vmatrix} \neq 0 \tag{2.4}$$

for all distinct $i_1, ..., i_{m+1} \in \{1, ..., n\}$.

We refer to the components $q_1, ..., q_n$ of a point $q = (q_1, ..., q_n)$ in $\mathcal{E}^n(\mathbb{CP}^m, gp)$ or in $\mathcal{E}^n(\mathbb{C}^m, gp)$ as to *vector coordinates* of q.

Although we use the same notation for determinant polynomials in both cases, projective and affine, every time we meet them it will be clear from the context which one we mean.

Notation 2.1. By a multiindex we mean an ordered set $\mathbf{i} = (i_1, ..., i_s)$ with distinct $i_1, ..., i_s \in \{1, ..., n\}$. Sometimes we forget the order and write $i \in \mathbf{i}$ and $\#\,\mathbf{i} = s$. If $s = 1$ and $i \in \{1, ..., n\}$, we may write $\mathbf{i} = i$ instead of $\mathbf{i} = (i)$.

For t multiindices $\mathbf{i}_1 = (i_1^1, ..., i_s^1), ..., \mathbf{i}_t = (i_1^t, ..., i_s^t)$ such that $i_1^1, ..., i_s^1, ..., i_1^t, ..., i_s^t$ are distinct, we set $d_{\mathbf{i}_1,...,\mathbf{i}_t} = d_{i_1^1,...,i_s^1,...,i_1^t,...,i_s^t}$.

Let $\mathbf{i} = (i_1, ..., i_s)$ and $\mathbf{i}' = (i_1, ..., i_{s-1})$; for any $i \in \{i_1, ..., i_s\}$ and any $k = 1, ..., s-1$, we denote by $D_{\mathbf{i};i,k}$ the $(s-1) \times (s-1)$ minor of the $s \times s$ matrix

$$Z_{\mathbf{i}} = \begin{pmatrix} z_{i_1,1}, ..., z_{i_1,s-1}, 1 \\ \vdots \\ z_{i_s,1}, ..., z_{i_s,s-1}, 1 \end{pmatrix}$$

complementary to the elements $z_{i,k}$; for $i = i_s$, we write $\delta_{\mathbf{i}';k}$ instead of $D_{\mathbf{i};i_s,k}$.

Let $\mathbf{I}^s$ denote the set of all multiindices $\mathbf{i} = (i_1, ..., i_s)$ such that $1 \leq i_1 < ... < i_s \leq n$. For $\mathbf{i} = (i_1, ..., i_s) \in \mathbf{I}^s$ and $\mathbf{j} = (j_1, ..., j_t) \in \mathbf{I}^t$, we define the multiindices $\mathbf{i} \cap \mathbf{j}$ and $\mathbf{i} \setminus \mathbf{j}$ in the evident way; if $\mathbf{i} \cap \mathbf{j} = \varnothing$, we define the multiindex $\mathbf{i} \cup \mathbf{j} \in \mathbf{I}^{s+t}$ by an appropriate reordering of the components $i_1, ..., i_s, j_1, ..., j_t$.

For $1 \leq j \leq n$, set $\mathbf{I}_j^s = \{\mathbf{i} \in \mathbf{I}^s \mid j \notin \mathbf{i}\}$. ○

2.2. **Irreducibility of determinant polynomials.**

LEMMA 2.2. *All minors of the matrices* (2.1) *and* (2.3) *are irreducible homogeneous polynomials in the entries* $z_{t,s}$.

Proof. It suffices to prove the lemma for the matrix (2.3). The proof is by induction in the order of a minor. Clearly, any minor of order 1 is an irreducible polynomial. Suppose that for some k, $1 \leq k < m+1$, all minors of order $\leq k$ are irreducible. By the Lagrange decomposition formula, any minor M of order $k+1$ is a linear function of the entries $z_{i_1,j_1}, ..., z_{i_{k+1},j_1}$ of its first column with coefficients that are certain minors of order k, i. e., polynomials of all entries of M but $z_{i_1,j_1}, ..., z_{i_{k+1},j_1}$. By the induction hypothesis, all the latter polynomials are irreducible; moreover, they depend on different sets of variables and hence cannot be proportional (with constant coefficients) to a single polynomial. This implies that M is irreducible. $\square$

LEMMA 2.3. *Let* $i_0 \in \{1, ..., n\}$, $\mathbf{i}_0 = (i_1, ..., i_m) \in \mathbf{I}_{i_0}^{m+1}$ *and* $L \subset \mathbb{C}^{mn}$ *be a linear subspace defined by the relations* $z_{i_1,2} = ... = z_{i_m,2}$.

a) For $\mathbf{i} \in \mathbf{I}_{i_0}^{m+1}$, *the restriction* $d_{\mathbf{i}}|_L$ *of* $d_{\mathbf{i}}$ *to* L *is irreducible if and only if* $\#(\mathbf{i} \cap \mathbf{i}_0) < m$. *Moreover, if* $\#(\mathbf{i} \cap \mathbf{i}_0) = m$, *then* $d_{\mathbf{i}}|_L = \pm D_{\mathbf{i};i,2} \cdot (z_{i,2} - z_{i_1,2})$, *where* $i \in \mathbf{i} \setminus \mathbf{i}_0$.

b) For $\mathbf{i} \in \mathbf{I}_{i_0}^{m+1}$, $\deg d_{\mathbf{i}}|_L = m$.

Proof. Assume that $\#(\mathbf{i} \cap \mathbf{i}_0) = m$. It follows that $\#(\mathbf{i} \setminus \mathbf{i}_0) = 1$. Set $(i) = \mathbf{i} \setminus \mathbf{i}_0$. By the Lagrange determinant decomposition formula, we can show that $d_{\mathbf{i}}|_L = \pm D_{\mathbf{i};i,2} \cdot (z_{i,2} - z_{i_1,2})$. That is, $d_{\mathbf{i}}|_L$ is reducible and $\deg d_{\mathbf{i}}|_L = m$.

Now assume that $\#(\mathbf{i} \cap \mathbf{i}_0) < m$. Let $i, j \in \mathbf{i} \setminus \mathbf{i}_0$ be two distinct indices. It is clear that if we prove that $d_{i,j,i_2,...,i_m}|_L$ is irreducible and $\deg d_{i,j,i_2,...,i_m}|_L = m$, the same statements for $d_{\mathbf{i}}|_L$ hold true. So set $\mathbf{j} = (i, j, i_2, ..., i_m)$. Obviously,

$$d_{\mathbf{j}}|_L = (z_{i_2,2} - z_{i,2}) \cdot D_{\mathbf{j};i,2} - (z_{i_2,2} - z_{j,2}) \cdot D_{\mathbf{j};j,2}$$

and $\deg d_{\mathbf{j}}|_L = m$. It follows that $d_{\mathbf{j}}|_L$ is a linear function of the variables $z_{i,2}, z_{j,2}$ and $z_{i_2,2}$ with coefficients $D_{\mathbf{j};i,2}, D_{\mathbf{j};j,2}$ and $D_{\mathbf{j};i,2} - D_{\mathbf{j};i,2}$. By Lemma 2.2, $D_{\mathbf{j};i,2}, D_{\mathbf{j};j,2}$ are irreducible. Therefore the polynomials $D_{\mathbf{j};i,2}, D_{\mathbf{j};j,2}$ and $D_{\mathbf{j};i,2} - D_{\mathbf{j};i,2}$ are pairwise co-prime. The latter implies that $d_{\mathbf{j}}|_L$ is irreducible. This completes the proof of the lemma. $\square$

2.3. **The direct decomposition of $\mathcal{E}^n(\mathbb{CP}^m, gp)$.**

Here we show that $\mathcal{E}^n(\mathbb{CP}^m, gp)$ admits a natural representation as a Cartesian product of its subspace of codimension $m(m+2)$ and the group $\mathbf{PSL}(m+1, \mathbb{C})$.

DEFINITION 2.4. Set

$$\begin{aligned} &v_1 = [1 : 0 : \cdots : 0]\,,\ v_2 = [0 : 1 : 0 : \cdots : 0]\,, ..., v_{m+1} = [0 : \cdots : 0 : 1] \\ &\text{and}\ \ w = [1 : \cdots : 1]\,. \end{aligned} \tag{2.5}$$

The subspace $M_{m,n} \subset \mathcal{E}^n(\mathbb{CP}^m, gp)$ defined by

$$M_{m,n} = \{q = (q_1, ..., q_n) \in \mathcal{E}^n(\mathbb{CP}^m, gp) |\ q_i = v_i\ \forall\, i = 1, ..., m+1\,,\ q_{m+2} = w\}$$

is called the *reduced space of geometrically generic ordered configurations*. $\bigcirc$

LEMMA 2.5. *Let* $n \geq m+3$. *For every* $q \in \mathcal{E}^n(\mathbb{CP}^m, gp)$, *there is a unique* $\gamma(q) \in \mathbf{PSL}(m+1, \mathbb{C})$ *such that* $\gamma(q)q \in M_{m,n}$. *The map*

$$\gamma\colon \mathcal{E}^n(\mathbb{CP}^m, gp) \ni q \mapsto \gamma(q) \in \mathbf{PSL}(m+1, \mathbb{C})$$

is holomorphic.

Proof. [3] For $q = \begin{pmatrix} [z_{1,1} : \dots : z_{1,m+1}] \\ \vdots \\ [z_{n,1} : \dots : z_{n,m+1}] \end{pmatrix} \in \mathcal{E}^n(\mathbb{CP}^m, gp)$ and $i \in \{1, ..., m+1\}$, the matrices

$$A(q) = \begin{pmatrix} z_{1,1} & ,\dots, & z_{1,m+1} \\ & \vdots & \\ z_{m+1,1} & ,\dots, & z_{m+1,m+1} \end{pmatrix}, \; A_i(q) = \begin{pmatrix} z_{1,1} & ,\dots, & z_{1,m+1} \\ & \vdots & \\ z_{i-1,1} & ,\dots, & z_{i-1,m+1} \\ z_{m+2,1} & ,\dots, & z_{m+2,m+1} \\ z_{i+1,1} & ,\dots, & z_{i+1,m+1} \\ & \vdots & \\ z_{m+1,1} & ,\dots, & z_{m+1,m+1} \end{pmatrix}$$

are defined up to multiplication of their rows by non-zero complex numbers. The determinants $\det A(q)$ and all $D_i = \det A_i$ are homogeneous polynomials non-vanishing on $\mathcal{E}^n(\mathbb{CP}^m, gp)$; therefore, the determinant of the adjunct matrix $X(q) = A^{adj}(q)$ (whose elements $x_{i,j}$ are the algebraic co-factors of the elements $z_{j,i}$ of $A(q)$) is equal to $(\det A(q))^m$ and hence does not vanish on $\mathcal{E}^n(\mathbb{CP}^m, gp)$. It follows that for every $q \in \mathcal{E}^n(\mathbb{CP}^m, gp)$ the matrix $T(q) = (x_{i,j}/D_j)_{i,j=1}^{m+1}$ is non-degenerate. It is easy to verify that $T(q)$ determines a well-defined automorphism $\gamma(q)$ of $\mathbb{CP}^m$ and $\gamma(q)q \in M_{m,n}$.

An automorphism of $\mathbb{CP}^m$ is uniquely determined by its values at any $m+2$ points in geometrically generic position. It follows that for any $q \in \mathcal{E}^n(\mathbb{CP}^m, gp)$ the constructed above element $\gamma(q) \in \mathbf{PSL}(m+1, \mathbb{C})$ is the only element of $\mathbf{PSL}(m+1, \mathbb{C})$ that carries q to a point of $M_{m,n}$. Moreover, $\gamma(q)$ holomorphically depends on a point q. □

Corollary 2.6. *The mutually inverse maps*

$$A\colon \mathcal{E}^n(\mathbb{CP}^m, gp) \ni q \mapsto A(q) = (\gamma(q), \gamma(q)q) \in \mathbf{PSL}(m+1, \mathbb{C}) \times M_{m,n}$$

and

$$B\colon \mathbf{PSL}(m+1, \mathbb{C}) \times M_{m,n} \ni (T, \tilde{q}) \mapsto B(T, \tilde{q}) = T^{-1}\tilde{q} \in \mathcal{E}^n(\mathbb{CP}^m, gp)$$

induce a natural biholomorphic isomorphism $\mathcal{E}^n(\mathbb{CP}^m, gp) \cong \mathbf{PSL}(m+1, \mathbb{C}) \times M_{m,n}$.

Remark 2.7. The above corollary implies that $\mathcal{E}^n(\mathbb{CP}^m, gp)$ and $\mathcal{C}^n(\mathbb{CP}^m, gp)$ are irreducible non-singular complex affine algebraic varieties and, in particular, Stein manifolds. Indeed, in the above direct decomposition of $\mathcal{E}^n(\mathbb{CP}^m, gp)$ both $\mathbf{PSL}(m+1, \mathbb{C})$ and $M_{m,n}$ are such varieties and hence $\mathcal{E}^n(\mathbb{CP}^m, gp)$ is also of the same nature. Since the group $\mathbf{S}(n)$ is finite and its action on $\mathcal{E}^n(\mathbb{CP}^m, gp)$ is free, the same is true for the quotient $\mathcal{C}^n(\mathbb{CP}^m, gp) = \mathcal{E}^n(\mathbb{CP}^m, gp)/\mathbf{S}(n)$.

The same properties hold true for $\mathcal{E}^n(\mathbb{C}^m, gp)$ and $\mathcal{C}^n(\mathbb{C}^m, gp)$.

Notice that for $n > 1$ the "standard" n-point configuration spaces $\mathcal{E}^n(X)$ and $\mathcal{C}^n(X)$ of a complex manifold X of dimension $m > 1$ cannot be Stein manifolds. ◯

[3]For $m = 1$ the statement of Lemma is a common knowledge; the case $m = 2$ is treated in [2], Chap. V, Sec. 109, Theorem 36. For $m > 2$, I could not find an appropriate reference and sketched the proof here.

2.4. **Determinant cross ratios.** Here we construct certain non-constant holomorphic functions $\mathcal{E}^n \to \mathbb{C} \setminus \{0,1\}$, which are called "determinant cross ratios" (in fact, later we shall show that there are no other functions with these properties).

Let us recall that, according to Notation 2.1, for distinct $i_1, ..., i_{m-1}, j, k$ and $\mathbf{i} = (i_1, ..., i_{m-1})$ the notation $d_{\mathbf{i},j,k}$ means the determinant $d_{i_1,...,i_{m-1},j,k}$.

Definition 2.8. Let X be either $\mathbb{CP}^m$ or $\mathbb{C}^m$ and let $n \geq m+3$. For any $m+3$-dimensional multiindex $I = (i_1, ..., i_{m+3})$ with distinct components $i_t \in \{1, ..., n\}$, set $\mathbf{i} = (i_1, ..., i_{m-1})$, $j = i_m$, $k = i_{m+1}$, $l = i_{m+2}$, $s = i_{m+3}$; the non-constant rational function

$$e_I(q) = e_{\mathbf{i};j,k,l,s}(q) = \frac{d_{\mathbf{i},j,k}(q)}{d_{\mathbf{i},j,l}(q)} : \frac{d_{\mathbf{i},k,s}(q)}{d_{\mathbf{i},l,s}(q)}\,, \quad q \in (\mathbb{CP}^m)^n\,, \tag{2.6}$$

is called a *determinant cross ratio*, or, in brief, a DCR. This function is regular on the algebraic manifold $\mathcal{E}^n(X, gp) \subset (\mathbb{CP}^m)^n$.

The unordered set of indices $\{I\} = \{i_1, \ldots, i_{m-1}, j, k, l, s\}$ is called the *support* of the function $\mu = e_I = e_{\mathbf{i};j,k,l,s}$ and is denoted by $\operatorname{supp}\mu$; its unordered subset $\{\mathbf{i}\} = \{i_1, ..., i_{m-1}\}$ is called the *essential support* of μ and is denoted by $\operatorname{supp}_{\text{ess}}\mu$ (we often write I instead of $\{I\}$ and $\mathbf{i}$ instead of $\{\mathbf{i}\}$). In fact, the function $e_I(q) = e_I(q_1, ..., q_n)$ depends only on the vector variables q_t with $t \in I$. ◯

Remark 2.9. Notice that two determinant cross ratios, say $e_I = e_{\mathbf{i};j,k,l,s}$ and $e_{I'} = e_{\mathbf{i}';j',k',l',s'}$, coincide if and only if $\{\mathbf{i}\} = \{\mathbf{i}'\}$ and (j', k', l', s') is obtained from (j, k, l, s) by a Kleinian permutation of four letters. The set of all determinant cross ratios is denoted by $\mathrm{DCR}(\mathcal{E}^n) = \mathrm{DCR}(\mathcal{E}^n(X, gp))$. ◯

The $\mathbf{S}(n)$ action in $\mathcal{E}^n$ induces an $\mathbf{S}(n)$ action on functions defined by $(\sigma\lambda)(q) = \lambda(\sigma^{-1}q) = \lambda(q_{\sigma(1)}, ..., q_{\sigma(n)})$ (λ is a function on $\mathcal{E}^n$, $q = (q_1, ..., q_n) \in \mathcal{E}^n$; notice that $\sigma\lambda$ may also be written as $(\sigma^{-1})^*\lambda = \lambda \circ (\sigma^{-1})$, where σ and σ^{-1} are considered as the self-mappings of $\mathcal{E}^n$). This action carries holomorphic functions to holomorphic functions.

Lemma 2.10. *For any $\sigma \in \mathbf{S}(n)$ and any $\lambda \in \mathrm{DCR}(\mathcal{E}^n)$ the function $\sigma\lambda$ also belongs to $\mathrm{DCR}(\mathcal{E}^n)$. Moreover, the $\mathbf{S}(n)$ action is transitive on the set $\mathrm{DCR}(\mathcal{E}^n)$.*

Proof. Let $\mathbf{a} = (a_1, ..., a_l)$ be a multiindex. For any $\sigma \in \mathbf{S}(n)$, let $\sigma(\mathbf{a})$ denote the multiindex $(\sigma(a_1), ..., \sigma(a_l))$. For any $\sigma \in \mathbf{S}(n)$, we have $\sigma d_{\mathbf{a}}(q) = d_{\mathbf{a}}(\sigma^{-1}q) = d_{\sigma(\mathbf{a})}(q)$; thus,

$$\begin{aligned}\sigma e_{\mathbf{i};j,k,r,s}(q) = e_{\mathbf{i};j,k,r,s}(\sigma^{-1}q) &= \frac{d_{\mathbf{i},j,k}(\sigma^{-1}q)}{d_{\mathbf{i},j,r}(\sigma^{-1}q)} : \frac{d_{\mathbf{i},k,s}(\sigma^{-1}q)}{d_{\mathbf{i},r,s}(\sigma^{-1}q)} \\ &= \frac{d_{\sigma(\mathbf{i}),\sigma(j),\sigma(k)}(q)}{d_{\sigma(\mathbf{i}),\sigma(j),\sigma(r)}(q)} : \frac{d_{\sigma(\mathbf{i}),\sigma(k),\sigma(s)}(q)}{d_{\sigma(\mathbf{i}),\sigma(r),\sigma(s)}(q)} = e_{\sigma(\mathbf{i});\sigma(j),\sigma(k),\sigma(r),\sigma(s)}(q)\,.\end{aligned}$$

Let $e_I, e_{I'} \in \mathrm{DCR}(\mathcal{E}^n)$. Since each of the ordered sets I and I' consists of $m+3 \leq n$ distinct elements of the set $\{1, ..., n\}$, there is a permutation $\sigma \in \mathbf{S}(n)$ such that $\sigma I = I'$ and hence $\sigma e_I = e_{I'}$. □

Lemma 2.11. *Determinant cross ratios are invariants of the $\mathbf{PSL}(m+1, \mathbb{C})$ action on the configuration space $\mathcal{E}^n(\mathbb{CP}^m, gp)$.*

Proof. It is easy to observe that the following elementary operators do not change any determinant cross ratio:

$$[z_1 : \cdots : z_{m+1}] \mapsto [a_1 z_1 : \cdots : a_{m+1} z_{m+1}] \quad \text{ for } a_1 \cdot ... \cdot a_{m+1} \neq 0\,;$$
$$[z_1 : \cdots : z_i : \cdots : z_j : \cdots : z_{m+1}] \mapsto [z_1 : \cdots : z_j : \cdots : z_i : \cdots : z_{m+1}]\,;$$
$$[z_1 : \cdots : z_i : \cdots : z_j : \cdots : z_{m+1}] \mapsto [z_1 : \cdots : z_i + z_j : \cdots : z_j : \cdots : z_{m+1}]\,.$$

Any element of $\mathbf{PSL}(m+1, \mathbb{C})$ can be decomposed into a sequence of the elementary operators. This proves the lemma. □

LEMMA 2.12. *Any function $\lambda \in \mathrm{DCR}(\mathcal{E}^n)$ omits the values* 0 *and* 1.

Proof. By Lemma 2.11, any cross ratio λ is $\mathbf{PSL}(m+1, \mathbb{C})$ invariant. By Lemma 2.5, any orbit of $\mathbf{PSL}(m+1, \mathbb{C})$ action in $\mathcal{E}^n$ intersects the subspace $M_{m,n}$; hence, it sufices to show that λ does not accept the values 0 and 1 on $M_{m,n}$.

First, let $I = (1, ..., m+3)$, $\mathbf{i} = (1, ..., m-1)$ and $\lambda = e_I(q)$; then

$$\lambda(q) = e_I(q) = \frac{d_{\mathbf{i},m,m+1}(q)}{d_{\mathbf{i},m,m+2}(q)} : \frac{d_{\mathbf{i},m+1,m+3}(q)}{d_{\mathbf{i},m+2,m+3}(q)}\,.$$

For $q = (q_1, ..., q_n) \in M_{m,n}$ with $q_t = [z_{t,1} : \cdots : z_{t,m+1}]$, we have

$$d_{\mathbf{i},m,m+1}(q) = 1\,, \qquad d_{\mathbf{i},m,m+2}(q) = 1\,,$$
$$d_{\mathbf{i},m+1,m+3}(q) = -z_{m+3,m}\,, \qquad d_{\mathbf{i},m+2,m+3}(q) = z_{m+3,m+1} - z_{m+3,m}\,,$$

and hence

$$e_I(q) = 1 - \frac{z_{m+3,m+1}}{z_{m+3,m}}\,.$$

Since $d_{\mathbf{i},m,m+3}(q) \neq 0$, $d_{\mathbf{i},m+1,m+3}(q) \neq 0$ and $d_{\mathbf{i},m+2,m+3}(q) \neq 0$, we have $z_{m+3,m}, z_{m+3,m+1} \neq 0$ and $z_{m+3,m} \neq z_{m+3,m+1}$. Thus, $e_I(q) \neq 0, 1$ on $M_{m,n}$ and, consequently, on the whole of $\mathcal{E}^n$.

By Lemma 2.10, $\mathbf{S}(n)$ acts transitively on the set $\mathrm{DCR}(\mathcal{E}^n)$, which implies that any $\lambda \in \mathrm{DCR}(\mathcal{E}^n)$ does not accept the values 0 and 1. □

NOTATION 2.13. For $s \in \{1, ..., m\}$, set $\mathbf{m}(\hat{s}) = (1, ..., \hat{s}, ..., m)$. For $s = m$, we write sometimes $\widehat{\mathbf{m}}$ instead of $\mathbf{m}(\hat{m})$.

We have also the following lemma:

LEMMA 2.14. *a*) *The map $P\colon M_{m,n} \to (\mathbb{C}^{n-m-2})^m$ defined by*

$$q \mapsto P(q) = \begin{pmatrix} p_{1,m+3}(q), & ..., & p_{1,n}(q) \\ ... & ... & ... \\ p_{m,m+3}(q), & ..., & p_{m,n}(q) \end{pmatrix}\,,$$

with $p_{s,t}(q) = e_{\mathbf{m}(\hat{s});s,m+1,m+2,t}(q)$ for $s = 1, ..., m$ and $t = m+3, ..., n$, is a holomorphic embedding.
b) *$M_{m,n}$ is a hyperbolic space.*

Proof. a) For $q = (q_1, ..., q_n) \in M_{m,n}$ with $q_i = [z_{i,1} : \cdots : z_{i,m+1}]$ we have $z_{t,m+1} = d_{\widehat{\mathbf{m}},m,t}(q) \neq 0$ and $z_{t,s} = \pm d_{\mathbf{m}(\hat{s}),m+1,t}(q) \neq 0$. Furthermore,

$$d_{\mathbf{m}(\hat{s}),s,m+1}(q) = (-1)^{m-s}\,, \qquad d_{\mathbf{m}(\hat{s}),s,m+2}(q) = (-1)^{m-s}\,,$$
$$d_{\mathbf{m}(\hat{s}),m+1,t}(q) = -(-1)^{m-s} z_{t,m}\,, \qquad d_{\mathbf{m}(\hat{s}),m+2,t}(q) = (-1)^{m-s}(z_{t,m+1} - z_{t,m})\,;$$

thus,

$$p_{s,t}(q) = e_{\mathbf{m}(\hat{s});s,m+1,m+2,t}(q) = \frac{d_{\mathbf{m}(\hat{s}),s,m+1}(q)}{d_{\mathbf{m}(\hat{s}),s,m+2}(q)} : \frac{d_{\mathbf{m}(\hat{s}),m+1,t}(q)}{d_{\mathbf{m}(\hat{s}),m+2,t}(q)} = 1 - \frac{z_{t,m+1}}{z_{t,s}}\,.$$

If $q' = (q_1', ..., q_n') \in M_{m,n}$ with $q_i' = [z_{i,1}' : \cdots : z_{i,m+1}']$ and $p_{s,t}(q) = p_{s,t}(q')$ for $s = 1, ..., m$, then $z_{t,m+1} : z_{t,s} = z_{t,m+1}' : z_{t,s}'$ for all $s = 1, ..., m$ and hence $q_t = q_t'$. Thus, $P(q) = P(q')$ implies $q = q'$ and P is injective.

To see that P is an embedding, it suffices to observe that from the above calculation follows that at any point the Jacobi matrix of P is of maximal rank.

b) Since every determinant cross ratio omits values 0 and 1,

$$P(M_{m,n}) \subset (\mathbb{C} \setminus \{0,1\})^{m(n-m-2)}\,,$$

it follows that $M_{m,n}$ is a Kobayashi's hyperbolic complex manifold. □

Lemma 2.14 and Corollary 2.6 imply that the determinant cross ratios generate the whole algebra $\mathcal{A} = \mathbb{C}[\mathcal{E}^n(\mathbb{CP}^m, gp)]^{\mathbf{PSL}(m+1,\mathbb{C})} = \mathbb{C}[M_{m,n}]$ of invariants of the $\mathbf{PSL}(m+1,\mathbb{C})$ action on the algebraic variety $\mathcal{E}^n(\mathbb{CP}^m, gp)$. We shall see that the set DCR of all these generators coincides with the set $L(\mathcal{E}^n)$ of all non-constant holomorphic functions $\mathcal{E}^n \to \mathbb{C} \setminus \{0,1\}$ and that a strictly equivariant holomorphic map $f\colon \mathcal{E}^n(\mathbb{CP}^m, gp) \to \mathcal{E}^n(\mathbb{CP}^m, gp)$ induces a selfmap f^* of $L(\mathcal{E}^n)$; thereby, such a map f induces an endomorphism of the algebra $\mathcal{A}$. Together with Corollary 2.6, this provides an important information about f and eventually leads to the proof of Theorem 1.4.

3. Holomorphic functions omitting the values 0 and 1

In this section we describe explicitly all holomorphic functions $\mu\colon \mathcal{E}^n \to \mathbb{C} \setminus \{0,1\}$.

3.1. **ABC lemma.** The following lemma plays a crucial part in an explicit description of all holomorphic functions $\mathcal{E}^n(\mathbb{C}^m, gp) \to \mathbb{C} \setminus \{0,1\}$. Let us recall that, according to Notation 2.1, for distinct $i_1, ..., i_{m-1}, j, k$ and $\mathbf{i} = (i_1, ..., i_{m-1})$ the notation $d_{\mathbf{i},j,k}$ means the determinant $d_{i_1,...,i_{m-1},j,k}$.

Lemma 3.1. *Let $A, B, C \in \mathbb{C}[\mathbb{C}^{mn}] = \mathbb{C}[z_{1,1}, ..., z_{n,m}]$ be pairwise co-prime polynomials on $\mathbb{C}^{mn}$ non-vanishing on the configuration space $\mathcal{E}^n(\mathbb{C}^m, gp) \subset \mathbb{C}^{mn}$. Assume that at least one of them is non-constant and $A + B + C = 0$. Then there exist a multiindex $\mathbf{i} = (i_1, \ldots, i_{m-1})$, indices j, k, l, s, and a complex number $\alpha \neq 0$ such that all $i_1, \ldots, i_{m-1}, j, k, l, s$ are distinct and $A = \alpha d_{\mathbf{i},j,k} d_{\mathbf{i},l,s}$, $B = \alpha d_{\mathbf{i},j,l} d_{\mathbf{i},s,k}$ and $C = \alpha d_{\mathbf{i},j,s} d_{\mathbf{i},k,l}$.*

Proof. The polynomials A, B, C do not vanish on

$$\mathcal{E}^n(\mathbb{C}^m, gp) = \mathbb{C}^{mn} \setminus \bigcup_{\mathbf{i}} \{q \in \mathbb{C}^{mn} \mid d_{\mathbf{i}}(q) = 0\}\,.$$

It follows from Lemma 2.2 that

$$A = \alpha \prod_{\mathbf{i} \in \mathbf{I}^{m+1}} d_{\mathbf{i}}^{a_{\mathbf{i}}}\,, \quad B = \beta \prod_{\mathbf{i} \in \mathbf{I}^{m+1}} d_{\mathbf{i}}^{b_{\mathbf{i}}}\,, \quad C = \gamma \prod_{\mathbf{i} \in \mathbf{I}^{m+1}} d_{\mathbf{i}}^{c_{\mathbf{i}}}\,,$$

where $\alpha, \beta, \gamma \in \mathbb{C} \setminus \{0\}$ and $a_{\mathbf{i}}, b_{\mathbf{i}}, c_{\mathbf{i}} \in \mathbb{Z}_+$. The polynomials are homogeneous; thus, the equality $A + B + C = 0$ implies that $\deg A = \deg B = \deg C$; i.e.,

$\sum a_{\mathbf{i}} = \sum b_{\mathbf{i}} = \sum c_{\mathbf{i}}$. For every index i_0, $1 \le i_0 \le n$, we can write

$$A = A_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} d_{i_0,\mathbf{i}}^{a_{i_0,\mathbf{i}}}\,, \qquad B = B_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} d_{i_0,\mathbf{i}}^{b_{i_0,\mathbf{i}}}\,, \qquad C = C_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} d_{i_0,\mathbf{i}}^{c_{i_0,\mathbf{i}}}\,,$$

where $A_{i_0}, B_{i_0}, B_{i_0}$ are the products of all factors $d_{\mathbf{i}}$ that do not contain the variables $z_{i_0,1}, ..., z_{i_0,m}$, i. e.,

$$A_{i_0} = \pm\alpha \prod_{\mathbf{i}\in \mathbf{I}^{m+1}_{i_0}} d_{\mathbf{i}}^{a_{\mathbf{i}}}\,, \quad B_{i_0} = \pm\beta \prod_{\mathbf{i}\in \mathbf{I}^{m+1}_{i_0}} d_{\mathbf{i}}^{b_{\mathbf{i}}}\,, \quad C_{i_0} = \pm\gamma \prod_{\mathbf{i}\in \mathbf{I}^{m+1}_{i_0}} d_{\mathbf{i}}^{c_{\mathbf{i}}}\,. \tag{3.1}$$

Our main tool is induction on the dimension m. The proof is divided into two steps; the first one supplies us with a way of the reduction of the dimension, and the second one is the induction itself.

Step 1. Let us prove the following statement:

($*$) *There is an index t_0 such that the polynomials $A_{t_0}, B_{t_0}, C_{t_0}$ are constant.*

Proof. First, we prove that for any $i_0 \in \{1, ..., n\}$

$$\sum a_{i_0,\mathbf{i}} = \sum b_{i_0,\mathbf{i}} = \sum c_{i_0,\mathbf{i}} \text{ and } \deg A_{i_0} = \deg B_{i_0} = \deg C_{i_0}\,. \tag{3.2}$$

Without loss of generality, we can assume that either a) $\sum a_{i_0,\mathbf{i}} > \sum b_{i_0,\mathbf{i}} \ge \sum c_{i_0,\mathbf{i}}$ or b) $\sum a_{i_0,\mathbf{i}} = \sum b_{i_0,\mathbf{i}} > \sum c_{i_0,\mathbf{i}}$ or c) $\sum a_{i_0,\mathbf{i}} = \sum b_{i_0,\mathbf{i}} = \sum c_{i_0,\mathbf{i}}$.

Compare the terms of the maximal degree in the variable $z_{i_0,1}$ in the main equality $A+B+C=0$. In the case (a) we have $A_{i_0} \prod \delta_{\mathbf{i};1}^{a_{i_0,\mathbf{i}}} = 0$ (see Notation 2.1 for the definition of $\delta_{\mathbf{i};k}$). This means $A=0$, a contradiction. In the case (b) we obtain $A_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} \delta_{\mathbf{i};1}^{a_{i_0,\mathbf{i}}} + B_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} \delta_{\mathbf{i};1}^{b_{i_0,\mathbf{i}}} = 0$. By Lemma 2.2, we have[4]

$$\text{G. C. D.}(A_{i_0}, \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} \delta_{\mathbf{i};1}^{b_{i_0,\mathbf{i}}}) = \text{G. C. D.}(B_{i_0}, \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} \delta_{\mathbf{i};1}^{a_{i_0,\mathbf{i}}}) = 1\,.$$

It follows that $A_{i_0} + B_{i_0} = 0$ and $a_{i_0,\mathbf{i}} = b_{i_0,\mathbf{i}}$ for all $\mathbf{i} \in \mathbf{I}^m_{i_0}$. The latter implies that $C = -(A+B) = 0$, a contradiction. This completes the proof of (3.2).

Since $A+B+C=0$, it follows that for each $k=1,...,m$ the leading term of $A+B+C$ in the variable $z_{i_0,k}$ is 0. In other words, for each $k=1,...,m$, we obtain the following equality:

$$A_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} \delta_{\mathbf{i};k}^{a_{i_0,\mathbf{i}}} + B_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} \delta_{\mathbf{i};k}^{b_{i_0,\mathbf{i}}} + C_{i_0} \prod_{\mathbf{i}\in \mathbf{I}^m_{i_0}} \delta_{\mathbf{i};k}^{c_{i_0,\mathbf{i}}} = 0\,. \tag{3.3}$$

We may assume that there is an index i_0 such that $A_{i_0} \ne A$ and $A_{i_0} \ne \text{const}$; for otherwise, either $A = \text{const}$, which is a contradiction, or $A_{i_0} = \text{const}$ and ($*$) holds with $t_0 = i_0$. According to (3.2), for such i_0 we have $0 < \deg A_{i_0} = \deg B_{i_0} = \deg C_{i_0} < \deg A = \deg B = \deg C$.

By Lemma 2.2, the polynomials $\delta_{\mathbf{i};k}$ are irreducible and distinct. Since A, B, C are pairwise co-prime, the integers $a_{i_0,\mathbf{i}}, b_{i_0,\mathbf{i}}, c_{i_0,\mathbf{i}}$ are also distinct. Using these facts, it easy to verify that the second order system of the linear equations in variables $A_{i_0}, B_{i_0}, C_{i_0}$ defined by (3.3) with $k=1,2$ is of rank 2; actually, all its 2×2 minors

[4] We denote the greatest common divisor by G. C. D..

are non-zero polynomials. It follows that $\dfrac{A_{i_0}}{\mathfrak{A}} = \dfrac{B_{i_0}}{\mathfrak{B}} = \dfrac{C_{i_0}}{\mathfrak{C}}$ with certain non-zero polynomials $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$. The polynomials B_{i_0} and C_{i_0} are co-prime, consequently,

$$\mathfrak{B} = \tilde{B} \cdot B_{i_0} \quad \text{and} \quad \mathfrak{C} = \tilde{C} \cdot C_{i_0}\,, \tag{3.4}$$

where $\tilde{B}, \tilde{C}$ are non-zero polynomials. Pick a multiindex $\mathbf{i}_0 = (i_1, \ldots, i_m) \in \mathbf{I}^m_{i_0}$ such that $a_{i_0,\mathbf{i}_0} > 0$. Since the polynomials A, B, C are pairwise co-prime, it follows that $b_{i_0,\mathbf{i}_0} = c_{i_0,\mathbf{i}_0} = 0$.

Let us prove that

$$A = A_{i_0} d_{i_0,\mathbf{i}_0}^{a_{i_0,\mathbf{i}_0}}\,, \qquad B_{i_0} = \pm\beta \prod_{i \neq i_0} d_{i,\mathbf{i}_0}^{b_{i,\mathbf{i}_0}}\,, \qquad C_{i_0} = \pm\gamma \prod_{i \neq i_0} d_{i,\mathbf{i}_0}^{c_{i,\mathbf{i}_0}}\,, \tag{3.5}$$

the statement $(*)$ will follow from this by combinatorial considerations.

Let L be the linear subspace of $\mathbb{C}^{mn}$ defined by the relations $z_{i_1,2} = \ldots = z_{i_m,2}$. Since for $\mathbf{i} \in \mathbf{I}^m_{i_0}$ we have $\delta_{\mathbf{i};k}|_L = 0$ if and only if $\#(\mathbf{i} \cap \mathbf{i}_0) = m$ and $k \neq 2$, the restrictions

$$\mathfrak{B}|_L = -\left(\prod \delta_{\mathbf{i};2}^{a_{i_0,\mathbf{i}}} \cdot \prod \delta_{\mathbf{i};1}^{c_{i_0,\mathbf{i}}}\right)\Big|_L \quad \text{and} \quad \mathfrak{C}|_L = -\left(\prod \delta_{\mathbf{i};2}^{a_{i_0,\mathbf{i}}} \cdot \prod \delta_{\mathbf{i};1}^{b_{i_0,\mathbf{i}}}\right)\Big|_L \tag{3.6}$$

are non-zero polynomials. Thus $\tilde{B}|_L, \tilde{C}|_L \neq 0$. According to (3.6), the polynomials $\mathfrak{B}|_L$ and $\mathfrak{C}|_L$ are products of irreducible polynomials of degree $\leq (m-1)$. By (3.1),

$$B_{i_0}|_L = \pm\beta \prod_{\mathbf{i} \in \mathbf{I}^{m+1}_{i_0}} (d_{\mathbf{i}}|_L)^{b_{\mathbf{i}}} \qquad \text{and} \qquad C_{i_0}|_L = \pm\gamma \prod_{\mathbf{i} \in \mathbf{I}^{m+1}_{i_0}} (d_{\mathbf{i}}|_L)^{c_{\mathbf{i}}}\,. \tag{3.7}$$

By (3.4), $B_{i_0}|_L$ and $C_{i_0}|_L$ also must be products of irreducible polynomials of degree $\leq (m-1)$. By Lemma 2.3, this may happen if and only if the decompositions (3.7) of $B_{i_0}|_L$ and $C_{i_0}|_L$ contain only factors of the form $d_{i,\mathbf{i}_0}|_L$. I.e., we have showed that

$$B_{i_0} = \pm\beta \prod_{i \neq i_0} d_{i,\mathbf{i}_0}^{b_{i,\mathbf{i}_0}} \quad \text{and} \quad C_{i_0} = \pm\gamma \prod_{i \neq i_0} d_{i,\mathbf{i}_0}^{c_{i,\mathbf{i}_0}}\,.$$

To complete the proof of (3.5), we must show that $a_{i_0,\mathbf{i}_1} = 0$ for any $\mathbf{i}_1 \in \mathbf{I}^m_{i_0} \setminus \{\mathbf{i}_0\}$. Suppose on the contrary that there exists a multiindex $\mathbf{i}_1$ such that $a_{i_0,\mathbf{i}_1} > 0$ and $\mathbf{i}_0 \neq \mathbf{i}_1$. Then the same argument implies that

$$B_{i_0} = \pm\beta \prod_{i \neq i_0} d_{i,\mathbf{i}_0}^{b_{i,\mathbf{i}_0}} = \pm\beta' \prod_{i \neq i_0} d_{i,\mathbf{i}_1}^{b_{i,\mathbf{i}_1}}\,, \quad C_{i_0} = \pm\gamma \prod_{i \neq i_0} d_{i,\mathbf{i}_0}^{c_{i,\mathbf{i}_0}} = \pm\gamma' \prod_{i \neq i_0} d_{i,\mathbf{i}_1}^{c_{i,\mathbf{i}_1}}\,. \tag{3.8}$$

Since we assume that $B_{i_0}, C_{i_0} \neq \text{const}$, (3.8) can occur only if $B_{i_0} = \pm\beta d_{s,\mathbf{i}_0}^{b_{s,\mathbf{i}_0}}$ and $C_{i_0} = \pm\gamma d_{s,\mathbf{i}_0}^{c_{s,\mathbf{i}_0}}$. I.e., B_{i_0} and C_{i_0} have a non-trivial common factor, which contradicts our assumptions. Thus, we have proved (3.5).

By (3.2), the choice of i_0 implies that $\sum a_{i_0,\mathbf{i}} = \sum b_{i_0,\mathbf{i}} = \sum c_{i_0,\mathbf{i}} > 0$. Since $b_{i_0,\mathbf{i}_0} = 0$, the latter implies that there exists a multiindex $\mathbf{i}_2 \neq \mathbf{i}_0$ such that $b_{i_0,\mathbf{i}_2} > 0$. In the same way as we proved (3.5), we conclude that

$$A_{i_0} = \pm\alpha' \prod_{i \neq i_0} d_{i,\mathbf{i}_2}^{a_{i,\mathbf{i}_2}}\,, \qquad B = B_{i_0} d_{i_0,\mathbf{i}_2}^{b_{i_0,\mathbf{i}_2}}\,, \qquad C_{i_0} = \pm\gamma' \prod_{i \neq i_0} d_{i,\mathbf{i}_2}^{c_{i,\mathbf{i}_2}}\,. \tag{3.9}$$

Comparing (3.5) and (3.9), we conclude that $C_{i_0} = \pm\gamma d_{s,\mathbf{i}_0}^{c_{s,\mathbf{i}_0}}$ and $\mathbf{i}_0 \cap \mathbf{i}_2 \neq \varnothing$. Pick $t_0 \in \mathbf{i}_0 \cap \mathbf{i}_2$, then $A_{t_0} = \text{const}$. Due to (3.2), $\deg A_{i_0} = \deg B_{i_0} = \deg C_{i_0}$; that is, the polynomials $A_{t_0}, B_{t_0}, C_{t_0}$ are constant. This completes the proof of $(*)$.

Step 2. Let $i_0 \in \{1, ..., n\}$. Let us prove the following statement:

$(**)$ *Assume that the polynomials* $A_{i_0}, B_{i_0}, C_{i_0}$ *are constant. Then there exist a multiindex* $\mathbf{i} = (i_1, \ldots, i_{m-2})$*, indices* j, s, l, t*, and a complex number* $\alpha \neq 0$ *such that all* $i_0, i_1, \ldots, i_{m-2}, j, s, l, t$ *are distinct and* $A = \alpha d_{i_0,\mathbf{i},j,s} d_{i_0,\mathbf{i},l,t}$*,* $B = \alpha d_{i_0,\mathbf{i},j,l} d_{i_0,\mathbf{i},t,s}$ *and* $C = \alpha d_{i_0,\mathbf{i},j,t} d_{i_0,\mathbf{i},s,l}$.

Proof. Denote

$$A' = A_{i_0} \prod \delta_{\mathbf{i};1}^{a_{i_0,\mathbf{i}}}\,, \quad B' = B_{i_0} \prod \delta_{\mathbf{i};1}^{b_{i_0,\mathbf{i}}}\,, \quad C' = C_{i_0} \prod \delta_{\mathbf{i};1}^{c_{i_0,\mathbf{i}}}\,.$$

It is easily seen that A', B' and C' are pairwise co-prime on $\mathbb{C}^{(m-1)(n-1)}$ and do not vanish on the configuration space $\mathcal{E}^{n-1}(\mathbb{C}^{m-1}, gp) \subset \mathbb{C}^{(m-1)(n-1)}$. According to (3.3), $A' + B' + C' = 0$. The proof of $(**)$ is by induction on m.

Let $m = 2$. Lemma 5.1 of [11] states that for any three non-constant pairwise co-prime polynomials

$$P = \mathrm{a} \prod_{i \neq j} (x_i - x_j)^{a_{i,j}}\,, Q = \mathrm{b} \prod_{i \neq j} (x_i - x_j)^{b_{i,j}}\,, R = \mathrm{c} \prod_{i \neq j} (x_i - x_j)^{c_{i,j}}$$

in the variables $x_1, ..., x_n$ which satisfy the equation $A' + B' + C' = 0$, there exist distinct indices j, s, l, t and a complex number $\alpha \neq 0$ such that either

$$P = \alpha(x_j - x_s)\,,\ Q = \alpha(x_s - x_l)\,,\ R = \alpha(x_l - x_j)$$

or

$$P = \alpha(x_j - x_s)(x_l - x_t)\,,\ Q = \alpha(x_j - x_l)(x_t - x_s)\,,\ R = \alpha(x_j - x_t)(x_s - x_l)\,.$$

This lemma applies to the polynomials A', B', C' in the variables $z_{1,2}, ..., z_{n,2}$. It follows that there exist distinct indices j, s, l, t and a complex number $\alpha \neq 0$ such that either $A' = \alpha(z_{j,2} - z_{s,2})$, $B' = \alpha(z_{s,2} - z_{l,2})$, $C' = \alpha(z_{l,2} - z_{j,2})$ or $A' = \alpha(z_{j,2} - z_{s,2})(z_{l,2} - z_{t,2})$, $B' = \alpha(z_{j,2} - z_{l,2})(z_{t,2} - z_{s,2})$, $C' = \alpha(z_{j,2} - z_{t,2})(z_{s,2} - z_{l,2})$. Thus, for our original polynomials A, B, C we obtain that either

$$A = \alpha d_{i_0,j,s}, \quad B = \alpha d_{i_0,s,l}, \quad C = \alpha d_{i_0,l,j}$$

or

$$A = \alpha d_{i_0,j,s} d_{i_0,l,t}\,, \quad B = \alpha d_{i_0,j,l} d_{i_0,t,s}\,, \quad C = \alpha d_{i_0,j,t} d_{i_0,s,l}\,.$$

It is easily seen that in the first case

$$\begin{aligned} A + B + C &= d_{i_0,j,s} + d_{i_0,s,l} + d_{i_0,l,j} \\ &= z_{j,1} z_{s,2} - z_{s,1} z_{j,2} + z_{s,1} z_{l,2} - z_{l,1} z_{s,2} + z_{l,1} z_{j,2} - z_{j,1} z_{l,2} \not\equiv 0\,. \end{aligned}$$

Thus, the equality $A + B + C = 0$ can be fulfilled only in the second case, which provides the base of induction.

Suppose that $(**)$ is fulfilled for some $m = k-1 > 1$ and let us prove that then the same is true for $m = k$. Due to $(*)$ (from Step 1), the induction hypothesis applies to the polynomials A', B', C'; that is, there exist a multiindex $\mathbf{i} = (i_1, ..., i_{k-2})$, indices j, s, l, t, and a complex number $\alpha \neq 0$ such that all $i_1, ..., i_{k-2}, j, s, l, t$ are distinct and $A' = \alpha \cdot d_{\mathbf{i},j,s} \cdot d_{\mathbf{i},l,t}$, $B' = \alpha \cdot d_{\mathbf{i},j,l} \cdot d_{\mathbf{i},t,s}$ and $C' = \alpha \cdot d_{\mathbf{i},j,t} \cdot d_{\mathbf{i},s,l}$. I.e., the original polynomials A, B, C can be written as $A = \alpha \cdot d_{i_0,\mathbf{i},j,s} \cdot d_{i_0,\mathbf{i},l,t}$, $B = \alpha \cdot d_{i_0,\mathbf{i},j,l} \cdot d_{i_0,\mathbf{i},t,s}$ and $C = \alpha \cdot d_{i_0,\mathbf{i},j,t} \cdot d_{i_0,\mathbf{i},s,l}$. This completes the justification of $(**)$ and proves Lemma 3.1. $\square$

3.2. Explicit description of all holomorphic functions $\mathcal{E}^n \to \mathbb{C}\setminus\{0,1\}$.

NOTATION 3.2. For a complex space Z, we donote by $L(Z)$ the set of all non-constant holomorphic functions $Z \to \mathbb{C}\setminus\{0,1\}$.

THEOREM 3.3. *Let $X = \mathbb{C}^m$ or $X = \mathbb{CP}^m$. Then $L(\mathcal{E}^n(X, gp)) = \mathrm{DCR}(\mathcal{E}^n(X, gp))$*[5].

Proof. By Corollary 2.12, it suffices to prove the inclusion

$$L(\mathcal{E}^n(X, gp)) \subseteq \mathrm{DCR}(\mathcal{E}^n(X, gp))\,.$$

We follow [10]. Let $\mu \in L(\mathcal{E}^n(X, gp))$, that is, $\mu\colon \mathcal{E}^n(X, gp) \to \mathbb{C}\setminus\{0,1\}$ is a holomorphic function.

First, let $X = \mathbb{C}^m$. It follows from Big Picard Theorem that μ is a regular function on $\mathcal{E}^n(\mathbb{C}^m, gp)$; hence it is a rational function on $(\mathbb{C}^m)^n$ and there are co-prime polynomials $A, B \in \mathbb{C}[(\mathbb{C}^m)^n]$ that do not vanish on $\mathcal{E}^n(\mathbb{C}^m, gp)$ and such that $\mu = -A/B$. The function $1-\mu = (A+B)/B$ also omits the values 0, 1. The polynomials A, B and $C = -B - A$ are pairwise co-prime, do not vanish on $\mathcal{E}^n(\mathbb{C}^m, gp)$ and satisfy $A + B + C = 0$. Lemma 3.1 applies to the latter three polynomials and shows that $\mu = -A/B = -d_{\mathbf{i},j,k}d_{\mathbf{i},l,s}/d_{\mathbf{i},j,l}d_{\mathbf{i},s,k} = e_{\mathbf{i};j,k,l,s}$ for appropriate $\mathbf{i}, j, k, l, s$.

When $X = \mathbb{CP}^m$, we restrict μ from $\mathcal{E}^n(\mathbb{CP}^m, gp)$ to $\mathcal{E}^n(\mathbb{C}^m, gp)$ and apply the above result, which leads to the desired conclusion. □

The following lemma is a known fact of the classical invariant theory (for small dimension it was discovered by A. F. Möbius, [12], especially, Part 2). The proof may be extracted from [16], Section 2.14 (especially, Theorem 2.14.A). However, the exposition in [16] is rather complicated; for the reader's convenience, we give here an independent proof.

LEMMA 3.4. *Let $\mathbf{i} = (i_1, \ldots, i_{m-1})$ be a multiindex, and j, k, l, s be indices such that all $i_1, \ldots, i_{m-1}, j, k, l, s$ are distinct. Then $d_{\mathbf{i},j,k}d_{\mathbf{i},l,s} + d_{\mathbf{i},j,l}d_{\mathbf{i},s,k} + d_{\mathbf{i},j,s}d_{\mathbf{i},k,l} = 0$.*

Proof. By Lemma 2.12, the function $1 - e_{\mathbf{i};j,k,l,s}$ omits the values 0 and 1 on $\mathcal{E}^n$; thus, by Theorem 3.3, it is a determinant cross ratio, say $e_{\mathbf{i}';j',k',l',s'}$. That is,

$$1 - e_{\mathbf{i};j,k,l,s} = 1 - \frac{d_{\mathbf{i},j,k}d_{\mathbf{i},l,s}}{d_{\mathbf{i},j,l}d_{\mathbf{i},k,s}} = \frac{d_{\mathbf{i},j,l}d_{\mathbf{i},k,s} - d_{\mathbf{i},j,k}d_{\mathbf{i},l,s}}{d_{\mathbf{i},j,l}d_{\mathbf{i},k,s}} = e_{\mathbf{i}';j',k',l',s'} = \frac{d_{\mathbf{i}',j',k'}d_{\mathbf{i}',l',s'}}{d_{\mathbf{i}',j',l'}d_{\mathbf{i}',k',s'}}\,.$$

Since determinant polynomials are irreducible (see Lemma 2.2) and the polynomials $d_{\mathbf{i}',j',k'}d_{\mathbf{i}',l',s'}$ and $d_{\mathbf{i}',j',l'}d_{\mathbf{i}',k',s'}$ are co-prime, it follows that there is a complex number $c \neq 0$ such that

$$d_{\mathbf{i},j,l}d_{\mathbf{i},k,s} - d_{\mathbf{i},j,k}d_{\mathbf{i},l,s} = c\,d_{\mathbf{i}',j',k'}d_{\mathbf{i}',l',s'} \tag{3.10}$$

and

$$d_{\mathbf{i},j,l}d_{\mathbf{i},k,s} = c\,d_{\mathbf{i}',j',l'}d_{\mathbf{i}',k',s'}\,. \tag{3.11}$$

By definition of a determinant cross ratio and Lemma 2.2, the polynomials $d_{\mathbf{i},j,l}d_{\mathbf{i},k,s}$ and $d_{\mathbf{i},j,k}d_{\mathbf{i},l,s}$ are co-prime. From (3.10) it follows that $d_{\mathbf{i},j,l}d_{\mathbf{i},k,s}$, $d_{\mathbf{i},j,k}d_{\mathbf{i},l,s}$ and $d_{\mathbf{i}',j',k'}d_{\mathbf{i}',l',s'}$ are pairwise co-prime. By Lemma 3.1, equality (3.10) implies that

$$c\,d_{\mathbf{i}',j',k'}d_{\mathbf{i}',l',s'} = d_{\mathbf{i},j,s}d_{\mathbf{i},k,l}\,. \tag{3.12}$$

By (3.10), $d_{\mathbf{i},j,l}d_{\mathbf{i},k,s} - d_{\mathbf{i},j,k}d_{\mathbf{i},l,s} = d_{\mathbf{i},j,s}d_{\mathbf{i},k,l}$; that is,

$$d_{\mathbf{i},j,k}d_{\mathbf{i},l,s} + d_{\mathbf{i},j,l}d_{\mathbf{i},s,k} + d_{\mathbf{i},j,s}d_{\mathbf{i},k,l} = 0\,.$$

[5] See Definition 2.8.

□

COROLLARY 3.5. $e_{\mathbf{i};j,k,l,s} + e_{\mathbf{i};j,s,l,k} = 1$.

Proof. Indeed, by the above lemma,

$$\begin{aligned} e_{\mathbf{i};j,k,l,s}(q) + e_{\mathbf{i};j,s,l,k}(q) - 1 &= \frac{d_{\mathbf{i},j,k}(q)}{d_{\mathbf{i},j,l}(q)} : \frac{d_{\mathbf{i},k,s}(q)}{d_{\mathbf{i},l,s}(q)} + \frac{d_{\mathbf{i},j,s}(q)}{d_{\mathbf{i},j,l}(q)} : \frac{d_{\mathbf{i},s,k}(q)}{d_{\mathbf{i},l,k}(q)} - 1 \\ &= \frac{d_{\mathbf{i},j,k}(q)}{d_{\mathbf{i},j,l}(q)} : \frac{d_{\mathbf{i},k,s}(q)}{d_{\mathbf{i},l,s}(q)} + \frac{d_{\mathbf{i},j,s}(q)}{d_{\mathbf{i},j,l}(q)} : \frac{d_{\mathbf{i},k,s}(q)}{d_{\mathbf{i},k,l}(q)} - 1 \\ &= \frac{d_{\mathbf{i},j,k}(q)d_{\mathbf{i},l,s}(q) + d_{\mathbf{i},j,s}(q)d_{\mathbf{i},k,l}(q) + d_{\mathbf{i},j,l}(q)d_{\mathbf{i},s,k}(q)}{d_{\mathbf{i},j,l}(q)d_{\mathbf{i},k,s}(q)} \\ &= 0\,. \end{aligned}$$

□

In the following lemma we establish some simple relations between determinant cross ratios.

LEMMA 3.6. *a*) $e_{\mathbf{i};t,k,r,s} = e_{\mathbf{i};j,k,r,s}/e_{\mathbf{i};j,k,r,t}$;
b) $e_{\mathbf{i};j,k,t,s} = 1 - \dfrac{1 - e_{\mathbf{i};j,k,r,s}}{1 - e_{\mathbf{i};j,k,r,t}}$;
c) $e_{\mathbf{i};j,t,r,s} = \left(1 - \dfrac{1 - (e_{\mathbf{i};j,k,r,s})^{-1}}{1 - (e_{\mathbf{i};j,k,r,t})^{-1}}\right)^{-1}$;
d) $e_{\mathbf{j},i;j,k,r,s} = e_{\mathbf{j},j;i,k,r,s} e_{\mathbf{j},s;j,k,r,i} = e_{\mathbf{j},k;j,i,r,s} e_{\mathbf{j},r;j,k,i,s}$.

Proof. (a) follows directly form the definition of determinant cross ratios.
b) By Corollary 3.5,

$$\begin{aligned} 1 - e_{\mathbf{i};j,k,r,s} &= e_{\mathbf{i};j,s,r,k} = e_{\mathbf{i};r,k,j,s}\,, \\ 1 - e_{\mathbf{i};j,k,r,t} &= e_{\mathbf{i};r,k,j,t} = e_{\mathbf{i};r,k,j,t}\,. \end{aligned}$$

In view of (a) and Corollary 3.5, this implies

$$\frac{1 - e_{\mathbf{i};j,k,r,s}}{1 - e_{\mathbf{i};j,k,r,t}} = \frac{e_{\mathbf{i};r,k,j,s}}{e_{\mathbf{i};r,k,j,t}} = e_{\mathbf{i};t,k,j,s} = 1 - e_{\mathbf{i};j,k,t,s}\,.$$

c) Clearly, $(e_{\mathbf{i};j,k,r,s})^{-1} = e_{\mathbf{i};j,r,k,s}$ and $(e_{\mathbf{i};j,k,r,t})^{-1} = e_{\mathbf{i};j,r,k,t}$. By ($b$),

$$\left(1 - \frac{1 - (e_{\mathbf{i};j,k,r,s})^{-1}}{1 - (e_{\mathbf{i};j,k,r,t})^{-1}}\right) = 1 - \frac{1 - e_{\mathbf{i};j,r,k,s}}{1 - e_{\mathbf{i};j,r,k,t}} = e_{\mathbf{i};j,r,t,s}\,,$$

since $(e_{\mathbf{i};j,r,t,s})^{-1} = e_{\mathbf{i};j,t,r,s}$, this implies ($c$).
d) From the definition of a determinant cross ratio we have

$$\begin{aligned} e_{\mathbf{j},i;j,k,r,s}/e_{\mathbf{j},j;i,k,r,s} &= \frac{d_{\mathbf{j},i,j,k}}{d_{\mathbf{j},i,j,r}} \frac{d_{\mathbf{j},i,r,s}}{d_{\mathbf{j},i,k,s}} \frac{d_{\mathbf{j},j,i,r}}{d_{\mathbf{j},j,i,k}} \frac{d_{\mathbf{j},j,k,s}}{d_{\mathbf{j},j,r,s}} \\ &= \frac{d_{\mathbf{j},i,r,s}}{d_{\mathbf{j},i,k,s}} \frac{d_{\mathbf{j},j,k,s}}{d_{\mathbf{j},j,r,s}} = \frac{d_{\mathbf{j},s,j,k}}{d_{\mathbf{j},s,j,r}} \frac{d_{\mathbf{j},s,i,r}}{d_{\mathbf{j},s,i,k}} = e_{\mathbf{j},s;j,k,r,i}\,. \end{aligned} \tag{3.13}$$

Clearly, $e_{\mathbf{j},i;j,k,r,s} = e_{\mathbf{j},i;k,j,s,r}$ and $e_{\mathbf{j},k;j,i,r,s} = e_{\mathbf{j},k;i,j,s,r}$. By (3.13),

$$e_{\mathbf{j},i;j,k,r,s}/e_{\mathbf{j},k;j,i,r,s} = e_{\mathbf{j},i;k,j,s,r}/e_{\mathbf{j},k;i,j,s,r} = e_{\mathbf{j},r;k,j,s,i} = e_{\mathbf{j},r;j,k,i,s}\,.$$

□

4. Simplicial complex of holomorphic functions $\mathcal{E}^n \to \mathbb{C} \setminus \{0,1\}$

4.1. **Simplicial complex of holomorphic functions omitting two values.** It was shown in [11] that the set $L(Z)$ of all non-constant holomorphic functions $Z \to \mathbb{C} \setminus \{0,1\}$ on a complex space Z may be endowed with a natural structure of a simplicial complex $L_\vartriangle(Z)$ and the correspondence $Z \mapsto L_\vartriangle(Z)$ has some properties of a contravariant functor from the category of complex spaces and holomorphic mappings to the category of simplicial complexes and simplicial mappings. In this section, we apply this Lin's construction for $Z = \mathcal{E}^n$ and study some properties of the complex $L_\vartriangle(\mathcal{E}^n)$. First, we recall the definition of the complex $L_\vartriangle(Z)$.

Definition 4.1. Let Z be a complex space and $L(Z)$ be the set of all non-constant holomorphic functions $Z \to \mathbb{C} \setminus \{0,1\}$.

For $\mu, \nu \in L(Z)$, we say that ν is *a proper divisor of* μ and write $\nu \mid \mu$ if the quotient $\lambda = \mu : \nu \in L(Z)$, i. e., $\lambda \neq \mathrm{const}$ and $\lambda(z) \neq 1$ for all $z \in Z$; otherwise, we write $\nu \nmid \mu$. Clearly, $\nu \mid \mu$ is equivalent to $\mu \mid \nu$.

A non-empty ordered subset $\Delta = \{\mu_0, ..., \mu_s\} \subseteq L(Z)$ is said to be a simplex of dimension s with vertices $\mu_0, ..., \mu_s$ if $\mu_i \mid \mu_j$ for each pair of distinct i, j. Evidently, a non-empty subset of a simplex is a simplex, that is, we obtain a well-defined simplicial complex $L_\vartriangle(Z)$ with the set of vertices $L(Z)$.[6]

If Z is a quasi-projective algebraic variety, then the set $L(Z)$ consists of a finite number of regular functions and the complex $L_\vartriangle(Z)$ is finite (it can be empty).

A holomorphic map $f\colon Z \to Y$ of complex spaces induces the homomorphism $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(Z)$ of the algebras of holomorphic functions defined, as usual, by $f^*(\lambda) = \lambda \circ f$, $\lambda \in \mathcal{O}(Y)$. Let $\lambda \in L(Y)$; suppose that $f^*(\lambda) \neq \mathrm{const}$ (this is certainly the case whenever f is a dominant map of irreducible quasi-projective varieties). Then the map of the vertices $f^*\colon L(Y) \ni \lambda \mapsto \lambda \circ f \in L(Z)$ induces the simplicial map $f^*\colon L_\vartriangle(Y) \to L_\vartriangle(Z)$ whose restriction to each simplex $\Delta \subseteq L(Y)$ is injective and preserves dimensions of simplices. ○

Remark 4.2. Let $\mathbf{i} = (i_1, \ldots, i_{m-1})$ and

$$\mu = e_{\mathbf{i};j,k,s,t} = \frac{d_{\mathbf{i},j,k}}{d_{\mathbf{i},j,s}} : \frac{d_{\mathbf{i},k,t}}{d_{\mathbf{i},s,t}}\,.$$

Let $\mathbf{e}_1, ..., \mathbf{e}_m$ be the standard basis in $\mathbb{C}^m$, $\mathbf{u}_1 = 0$ and $\mathbf{u}_p = \sum\limits_{j=m-p+2}^{m} \mathbf{e}_j$ for $p = 2, ..., m-1$. Set

$$L = \{(v_1, ..., v_n) \in (\mathbb{C}^m)^n \,|v_{i_p} = \mathbf{u}_p \text{ for } p = 1, ..., m-1\}\,.$$

For $q = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in (\mathbb{C}^m)^n \cap L$ with all $q_r = (z_{r,1}, ..., z_{r,m}) \in \mathbb{C}^m$, the restriction of any determinant polynomial $d_{\mathbf{i},l,r}$ to the subspace L is reduced to a certain determinant of order 2 and may be computed as $(-1)^m(z_{l,1}z_{r,2} - z_{r,1}z_{l,2})$. Consequently, the restriction $\mu|_L$ of the function μ to the subspace L may be written as

$$\mu|_L(q) = \frac{(z_{j,1}/z_{j,2}) - (z_{k,1}/z_{k,2})}{(z_{j,1}/z_{j,2}) - (z_{s,1}/z_{s,2})} : \frac{(z_{k,1}/z_{k,2}) - (z_{t,1}/z_{t,2})}{(z_{s,1}/z_{s,2}) - (z_{t,1}/z_{t,2})}\,. \tag{4.1}$$

[6] It is well-known that if Z is a quasi-projective algebraic variety, then the set $L(Z)$ consists of a finite number of regular functions and the complex $L_\vartriangle(Z)$ is finite (it can be empty); however, we do not need the latter statement since we already know the explicit description of $L(\mathcal{E}^n)$.

Thus, $\mu|_L$ is the cross ratio of the four quantities $a = z_{j,1}/z_{j,2}$, $b = z_{k,1}/z_{k,2}$, $c = z_{j,1}/z_{j,2}$ and $d = z_{t,1}/z_{t,2}$, which may be treated as four distinct points in $\mathbb{CP}^1$ whenever $q \in \mathcal{E}^n \cap L$. If μ and μ' are determinant cross ratios, $\mu|\mu'$ and $\mathbf{i} = \operatorname{supp}_{\text{ess}} \mu = \operatorname{supp}_{\text{ess}} \mu'$, then $\operatorname{supp}_{\text{ess}}(\mu : \mu') = \operatorname{supp}_{\text{ess}} \mu = \operatorname{supp}_{\text{ess}} \mu'$. In view of the above, the restrictions of μ, μ' and $\mu : \mu'$ to L are usual cross ratios and, moreover, $(\mu|_L) : (\mu'|_L) = (\mu : \mu')|_L$. ◯

In view of the previous remark, certain results about usual cross ratios apply to the determinant cross ratios, as well. In particular, we will use the following simple lemma (compare to Lemma 5.7, [11]):

LEMMA 4.3. *If the ratio $\mu : \mu'$ of two cross ratios*

$$\mu = \frac{q_j - q_k}{q_j - q_s} : \frac{q_k - q_t}{q_s - q_t} \quad \text{and} \quad \mu' = \frac{q_{j'} - q_{k'}}{q_{j'} - q_{s'}} : \frac{q_{k'} - q_{t'}}{q_{s'} - q_{t'}}$$

is a cross ratio of certain four of the eight variables $q_j, q_k, q_s, q_t, q_{j'}, q_{k'}, q_{s'}, q_{t'}$ then $\#(\{j,k,s,t\} \cap \{j',k',s',t'\}) = 3$ and μ' is obtained from μ by replacing of one of the variables q_j, q_k, q_s, q_t with some q_m, where $m \neq j,k,s,t$. ◯

We need also the following technical lemma.

LEMMA 4.4. *Let X be either $\mathbb{C}^m$ or $\mathbb{CP}^m$, $n > m+2$, and let $\mu = e_{\mathbf{i};j,k,r,s}$ be a proper divisor of a determinant cross ratio μ'. Then $\#(\operatorname{supp}_{\text{ess}} \mu \cap \operatorname{supp}_{\text{ess}} \mu') \geq m-2$. If $\operatorname{supp} \mu' \neq \operatorname{supp} \mu$, then $\#(\operatorname{supp} \mu \cap \operatorname{supp} \mu') = m+2$, $\operatorname{supp}_{\text{ess}} \mu = \operatorname{supp}_{\text{ess}} \mu'$, and μ' is one of the functions $e_{\mathbf{i};j,k,r,t}$, $e_{\mathbf{i};k,j,s,t}$, $e_{\mathbf{i};r,s,j,t}$, $e_{\mathbf{i};s,r,k,t}$.*

Proof. Set

$$\mu = e_{\mathbf{i};j,k,r,s} = \frac{d_{\mathbf{i},j,k}}{d_{\mathbf{i},j,r}} : \frac{d_{\mathbf{i},k,s}}{d_{\mathbf{i},r,s}}, \quad \mu' = e_{\mathbf{i}';j',k',r',s'} = \frac{d_{\mathbf{i}',j',k'}}{d_{\mathbf{i}',j',r'}} : \frac{d_{\mathbf{i}',k',s'}}{d_{\mathbf{i}',r',s'}}.$$

Let $\mathbf{j} = \mathbf{i} \cap \mathbf{i}'$. Since $\mu|\mu'$, from Lemma 2.2 it follows that

$$\mathbf{i}, \mathbf{i}' \in (\operatorname{supp} \mu \cap \operatorname{supp} \mu') \subset \operatorname{supp} \mu\,. \tag{4.2}$$

Let us show that $\#\,\mathbf{j} \neq m-5, m-4, m-3$ and hence $\#\,\mathbf{j} \geq m-2$.

If $\#\,\mathbf{j} = m-5$, then by (4.2), $\mathbf{i} = (\mathbf{j}, j', k', r', s')$ and $\mathbf{i}' = (\mathbf{j}, j, k, r, s)$. Thus,

$$\mu = e_{\mathbf{j},j',k',r',s';j,k,r,s} = \frac{d_{\mathbf{j},j',k',r',s',j,k}}{d_{\mathbf{j},j',k',r',s',j,r}} : \frac{d_{\mathbf{j},j',k',r',s',k,s}}{d_{\mathbf{j},j',k',r',s',r,s}},$$
$$\mu' = e_{\mathbf{j},j,k,r,s;j',k',r',s'} = \frac{d_{\mathbf{j},j,k,r,s,j',k'}}{d_{\mathbf{j},j,k,r,s,j',r'}} : \frac{d_{\mathbf{j},j,k,r,s,k',s'}}{d_{\mathbf{j},j,k,r,s,r',s'}}.$$

Since μ/μ' is a determinant cross ratio, and determinant polynomials are irreducible (Lemma 2.2), $\#(\{j',k',r',s'\} \cap \{j,k,r,s\}) > 0$, and we come to $\#(\mathbf{i} \cap \mathbf{i}') > m-5$, which contradicts our assumption.

For $\#(\mathbf{i} \cap \mathbf{i}') = m-4$, in view of (4.2), we can assume that $\mathbf{i} = (\mathbf{j}, j', k', r')$ and $\mathbf{i}' = (\mathbf{j}, j, k, r)$. Thus,

$$\mu = e_{\mathbf{j},j',k',r';j,k,r,s} = \frac{d_{\mathbf{j},j',k',r',j,k}}{d_{\mathbf{j},j',k',r',j,r}} : \frac{d_{\mathbf{j},j',k',r',k,s}}{d_{\mathbf{j},j',k',r',r,s}},$$
$$\mu' = e_{\mathbf{j},j,k,r;j',k',r',s'} = \frac{d_{\mathbf{j},j,k,r,j',k'}}{d_{\mathbf{j},j,k,r,j',r'}} : \frac{d_{\mathbf{j},j,k,r,k',s'}}{d_{\mathbf{j},j,k,r,r',s'}}.$$

Since μ/μ' is a determinant cross ratio and determinant polynomials are irreducible (Lemma 2.2), $\#(\{j',k',r'\} \cap \{j,k,r\}) > 0$, a contradiction.

For $\#(\mathbf{i}\cap\mathbf{i}')=m-3$, in view of (4.2), we can assume that either

$$\mu = e_{\mathbf{j},j',k';j,k,r,s} = \frac{d_{\mathbf{j},j',k',j,k}}{d_{\mathbf{j},j',k',j,r}} : \frac{d_{\mathbf{j},j',k',k,s}}{d_{\mathbf{j},j',k',r,s}},$$
$$\mu' = e_{\mathbf{j},j,k;j',k',r',s'} = \frac{d_{\mathbf{j},j,k,j',k'}}{d_{\mathbf{j},j,k,j',r'}} : \frac{d_{\mathbf{j},j,k,k',s'}}{d_{\mathbf{j},j,k,r',s'}}$$

or

$$\mu = e_{\mathbf{j},j',k';j,k,r,s} = \frac{d_{\mathbf{j},j',k',j,k}}{d_{\mathbf{j},j',k',j,r}} : \frac{d_{\mathbf{j},j',k',k,s}}{d_{\mathbf{j},j',k',r,s}},$$
$$\mu' = e_{\mathbf{j},j,r;j',k',r',s'} = \frac{d_{\mathbf{j},j,r,j',k'}}{d_{\mathbf{j},j,r,j',r'}} : \frac{d_{\mathbf{j},j,r,k',s'}}{d_{\mathbf{j},j,r,r',s'}}$$

or

$$\mu = e_{\mathbf{j},j',k';j,k,r,s} = \frac{d_{\mathbf{j},j',k',j,k}}{d_{\mathbf{j},j',k',j,r}} : \frac{d_{\mathbf{j},j',k',k,s}}{d_{\mathbf{j},j',k',r,s}},$$
$$\mu' = e_{\mathbf{j},j,s;j',k',r',s'} = \frac{d_{\mathbf{j},j,s,j',k'}}{d_{\mathbf{j},j,s,j',r'}} : \frac{d_{\mathbf{j},j,s,k',s'}}{d_{\mathbf{j},j,s,r',s'}}$$

or

$$\mu = e_{\mathbf{j},j',s';j,k,r,s} = \frac{d_{\mathbf{j},j',s',j,k}}{d_{\mathbf{j},j',s',j,r}} : \frac{d_{\mathbf{j},j',s',k,s}}{d_{\mathbf{j},j',s',r,s}},$$
$$\mu' = e_{\mathbf{j},j,s;j',k',r',s'} = \frac{d_{\mathbf{j},j,s,j',k'}}{d_{\mathbf{j},j,s,j',r'}} : \frac{d_{\mathbf{j},j,s,k',s'}}{d_{\mathbf{j},j,s,r',s'}}.$$

Since μ/μ' is a determinant cross ratio and determinant polynomials are irreducible (Lemma 2.2), these cases are impossible, a contradiction. This completes the proof of the first part of the lemma.

Suppose now that $\operatorname{supp}\mu \neq \operatorname{supp}\mu'$. Since we have already proved that $m > \#\mathbf{j} \geq m-2$, we need to show that $\#\mathbf{j} \neq m-2$.

Suppose to the contrary that $\#\mathbf{j}=m-2$. Then, without loss of generality, we may assume that $\mu = e_{\mathbf{j},j';j,k,r,s}$ and $\mu' = e_{\mathbf{j},j;j',k',r',s'}$. Therefore,

$$\mu : \mu' = \left(\frac{d_{\mathbf{j},j',j,k}}{d_{\mathbf{j},j',j,r}} : \frac{d_{\mathbf{j},j',k,s}}{d_{\mathbf{j},j',r,s}}\right) : \left(\frac{d_{\mathbf{j},j,j',k'}}{d_{\mathbf{j},j,j',r'}} : \frac{d_{\mathbf{j},j,k',s'}}{d_{\mathbf{j},j,r',s'}}\right)$$
$$= \frac{d_{\mathbf{j},j',j,k}d_{\mathbf{j},j',r,s}d_{\mathbf{j},j,j',r'}d_{\mathbf{j},j,k',s'}}{d_{\mathbf{j},j,j',k'}d_{\mathbf{j},j',k,s}d_{\mathbf{j},j',j,r}d_{\mathbf{j},j,r',s'}}.$$

Since all determinant polynomials are irreducible (Lemma 2.2), the latter quotient is a determinant cross ratio if and only if $k'=k$, $r'=r$ and $s'=s$, which may happen if and only if $\operatorname{supp}\mu = \operatorname{supp}\mu'$, a contradiction.

We are left with the case $\#\mathbf{j}=m-1$, that is, $\mathbf{i}=\mathbf{i}'$. Let L be as in Remark 4.2; then the restrictions $\mu|_L$ and $\mu'|_L$ of μ and μ' to L are usual cross ratios of the variables p_j,p_k,p_r,p_s and $p_{j'},p_{k'},p_{r'},p_{s'}$, respectively, and $(\mu|_L):(\mu'|_L)$ is such a cross ratio, as well. By Lemma 4.3, $\#(\{j,k,r,s\}\cap\{j',k',r',s'\})=3$ and the ordered set $\{j',k',r',s'\}$ is obtained by replacing of one of the indices in the ordered set $\{j,k,r,s\}$ with some index t, where $t\neq j,k,r,s$ (up to a Kleinian permutation). Consequently, $\#(\operatorname{supp}\mu\cap\operatorname{supp}\mu')=\#\mathbf{i}+3=m+2$ and μ' is one of the functions $e_{\mathbf{i};j,k,r,t}$, $e_{\mathbf{i};k,j,s,t}$, $e_{\mathbf{i};r,s,j,t}$, $e_{\mathbf{i};s,r,k,t}$. □

4.2. **$\mathbf{S}(n)$ action in $L_\vartriangle(\mathcal{E}^n)$.** The $\mathbf{S}(n)$ action in $\mathcal{E}^n$ induces an $\mathbf{S}(n)$ action on the set $L(\mathcal{E}^n)$ of all non-constant holomorphic functions $\mathcal{E}^n \to \mathbb{C} \setminus \{0,1\}$. Of course, this is the action on the set of all determinant cross ratios that we dealt with in Lemma 2.10; thus, it is transitive.

If $\mu, \nu \in L(\mathcal{E}^n)$ and $\sigma \in \mathbf{S}(n)$, then the relations $\mu|\nu$ and $(\sigma\mu)|(\sigma\nu)$ are equivalent. Therefore, the above action induces a simplicial dimension preserving $\mathbf{S}(n)$ action on the complex $L_\vartriangle(\mathcal{E}^n)$; that is, $\mathbf{S}(n)$ acts on the set of all simplices of any fixed dimension s. Our nearest goal is to describe the orbits of the latter action; in particular, we shall prove that on the set of all simplices of any positive dimension it has exactly two orbits.

According to Notation 2.13, $\mathbf{m}(\hat{s}) = (1, ..., \hat{s}, ..., m)$; sometimes we write $\widehat{\mathbf{m}}$ instead of $\mathbf{m}(\hat{m})$.

Definition 4.5. The t-simplex

$$\nabla_1^t = \{e_{\widehat{\mathbf{m}};m,m+1,m+2,m+3}; e_{\widehat{\mathbf{m}};m,m+1,m+2,m+4}; ...; e_{\widehat{\mathbf{m}};m,m+1,m+2,m+3+t}\}$$

is called the *normal t-simplex of the first type*; such simplices do exist for $0 \le t \le n-m-3$. We say that a t-simplex is of the *first type* if it belongs to the $\mathbf{S}(n)$ orbit of the normal simplex ∇_1^t.

The t-simplex

$$\nabla_2^t = \{e_{\mathbf{m}(\widehat{m-t});m-t,m+1,m+2,m+3}; e_{\mathbf{m}(\widehat{m-t+1});m-t+1,m+1,m+2,m+3}; \\ ...; e_{\mathbf{m}(\hat{m});m,m+1,m+2,m+3}\}$$

is called the *normal t-simplex of the second type*; such simplices do exist for $0 \le t \le m-1$. We say that a t-simplex is of the *second type* if it belongs to the $\mathbf{S}(n)$ orbit of the normal simplex ∇_2^t.

Notice that for $t > 0$ the simplices ∇_1^t and ∇_2^t belong to different orbits of the $\mathbf{S}(n)$ action on the set of all t-simpices. ◯

Remark 4.6. If $\Delta_1 = \{\mu_0, ..., \mu_t\}$ and $\Delta_2 = \{\nu_0, ..., \nu_t\}$ are simplices of the same type and the corresponding sets of functions μ's and ν's both involve only the vector variables $q_{i_1}, ..., q_{i_r}$, then Δ_1 may be carried to Δ_2 by a permutation $\sigma \in \mathbf{S}(\{i_1, ..., i_r\}) \subset \mathbf{S}(n)$. ◯

The following lemma shows that the $\mathbf{S}(n)$ action on the set of all simplices of dimension $t > 0$ has at most two orbits and each of these orbits consists of all simplices of the same type. Notice that, by Theorem 3.3, any vertex of the complex $L_\vartriangle(\mathcal{E}^n)$ is a determinant cross ratio.

Lemma 4.7. *Let X be either $\mathbb{C}^m$ or $\mathbb{CP}^m$ and $n > m+2$.*

a) Let $\Delta = \{\mu_s\}_{s=1}^{l+1} \in L_\vartriangle(\mathcal{E}^n)$ be an l-dimensional simplex and let $\operatorname{supp}\mu_1 = \operatorname{supp}\mu_2 = ... = \operatorname{supp}\mu_{l+1}$. *Then $\#(\operatorname{supp}_{\text{ess}}\mu_s \cap \operatorname{supp}_{\text{ess}}\mu_t) = m-2$ for all $t \ne s$ and $l \le m-1$. Moreover, the simplex Δ is of the second type.*

b) Let $\Delta = \{\mu_s\}_{s=1}^{l+1} \in L_\vartriangle(\mathcal{E}^n)$ be an l-dimensional simplex. If $\operatorname{supp}\mu_{s_0} \ne \operatorname{supp}\mu_{t_0}$ *for some $s_0 \ne t_0$, then* $\operatorname{supp}\mu_s \ne \operatorname{supp}\mu_t$ *and* $\operatorname{supp}_{\text{ess}}\mu_s = \operatorname{supp}_{\text{ess}}\mu_t$ *for all $s \ne t$. Moreover, the simplex Δ is of the first type.*

c) $\dim L_\vartriangle(\mathcal{E}^n(X)) = \max\{n-(m+3), m-1\}$.

Proof. a) Suppose, to the contrary, that $\#(\operatorname{supp}_{\text{ess}}\mu_s \cap \operatorname{supp}_{\text{ess}}\mu_t) \ne m-2$ for certain $s \ne t$. By Lemma 4.4, this means that $\#(\operatorname{supp}_{\text{ess}}\mu_s \cap \operatorname{supp}_{\text{ess}}\mu_t) > m-2$. Since $\#\operatorname{supp}_{\text{ess}}\mu_s = \#\operatorname{supp}_{\text{ess}}\mu_t = m-1$, it follows that $\operatorname{supp}_{\text{ess}}\mu_s = \operatorname{supp}_{\text{ess}}\mu_t$;

endow the latter set with some order and denote it by $\mathbf{i} = (i_1, ..., i_{m-1})$. Let L be as in Remark 4.2; then the restrictions $\mu_s|_L$ and $\mu_t|_L$ of μ_s and μ_t to L are usual cross ratios of the variables p_j, p_k, p_s, p_t and $p_{j'}, p_{k'}, p_{s'}, p_{t'}$, respectively, and $(\mu_s|_L) : (\mu_t|_L)$ is such a cross ratio, as well. By Lemma 4.3, $\#(\{j,k,s,t\} \cap \{j',k',s',t'\}) = 3$. Consequently, $\#(\operatorname{supp} \mu_s \cap \operatorname{supp} \mu_t) = \# \operatorname{supp}_{\mathrm{ess}} \mu_s + 3 = m+2$. Since $\# \operatorname{supp} \mu_s = \# \operatorname{supp} \mu_t = m+3$, it follows that $\operatorname{supp} \mu_s \neq \operatorname{supp} \mu_t$, a contradiction. Hence, for $s \neq t$ we have $\#(\operatorname{supp}_{\mathrm{ess}} \mu_s \cap \operatorname{supp}_{\mathrm{ess}} \mu_t) = m-2$.

Now let us show that $l \leq m-1$.

Let $\mathbf{j} = (i_1, ..., i_{m-2}) = \operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_2$; then $\mu_1 = e_{\mathbf{j},i;j,k,r,s}$ with certain i,j,k,r,s. By Lemma 2.2 and a straightforward computation, one can show that $\mu_2 \in D = \{e_{\mathbf{j},j;i,k,r,s}, e_{\mathbf{j},k;j,i,r,s}, e_{\mathbf{j},r;j,k,i,s}, e_{\mathbf{j},s;j,k,r,i}\}$; the latter set contains no pair of functions $\{\nu, \nu'\}$ that are vertices of the same simplex.

If $m=2$, then $\mathbf{j} = \varnothing$ and we must have $l \leq 1$, for otherwise it is easy to show that $\mu_3 \in D$ and μ_2 could not be a proper divisor of μ_3.

Assume now that $m > 2$. Then $\#(\operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_2 \cap \operatorname{supp}_{\mathrm{ess}} \mu_t) < m-2$ for any $t > 2$, since the equality $\#(\operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_2 \cap \operatorname{supp}_{\mathrm{ess}} \mu_t) = m-2$ would imply $\mathbf{j} = \operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_2 = \operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_t$ and $\mu_t \in D$, which is impossible. Since the intersection of any two of the three $m-1$ point sets $\operatorname{supp}_{\mathrm{ess}} \mu_1$, $\operatorname{supp}_{\mathrm{ess}} \mu_2$ and $\operatorname{supp}_{\mathrm{ess}} \mu_t$ consists of $m-2$ points, the intersection of all of them contains at least $m-3$ points. Thus, $\#(\operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_2 \cap \operatorname{supp}_{\mathrm{ess}} \mu_t) = m-3$.

Furthermore, for any $t > 2$, there is a unique $i' \in \operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_2$ such that $i' \notin \operatorname{supp}_{\mathrm{ess}} \mu_t$. We shall show that μ_t is uniquely determined by this i'.

Let $\mathbf{i} = (i_1, ..., i_{m-3}) = \operatorname{supp}_{\mathrm{ess}} \mu_1 \cap \operatorname{supp}_{\mathrm{ess}} \mu_2 \cap \operatorname{supp}_{\mathrm{ess}} \mu_t$ and $\mu_1 = e_{\mathbf{i},i,i';j,k,r,s}$. By Lemma 2.2 and a straightforward computation, one can show that μ_2 belongs to the set

$$S = \{e_{\mathbf{i},i',j;i,k,r,s}, e_{\mathbf{i},i',k;j,i,r,s}, e_{\mathbf{i},i',r;j,k,i,s}, e_{\mathbf{i},i',s;j,k,r,i}\}\,.$$

Similarly,

$$\mu_t \in T = \{e_{\mathbf{i},i,j;i',k,r,s}, e_{\mathbf{i},i,k;j,i',r,s}, e_{\mathbf{i},i,r;j,k,i',s}, e_{\mathbf{i},i,s;j,k,r,i'}\}\,.$$

Neither S nor T contain a pair of functions $\{\nu, \nu'\}$ such that $\nu|\nu'$. Since

$$\#(\operatorname{supp}_{\mathrm{ess}} \mu_2 \cap \operatorname{supp}_{\mathrm{ess}} \mu_t) = m-2\,,$$

for every $\nu \in S$ there is only one $\nu' \in T$ such that $\nu|\nu'$; this shows that μ_t is uniquely determined by i'. It follows that $l = \dim \Delta \leq m-1$.

Finally, in view of the above facts and the transitivity of the $\mathbf{S}(n)$ action on 0 dimensional simplices (see Lemma 2.10), the last statement of the part (a) of the lemma is obvious.

$b)$ By Lemma 4.4, for $l < 2$ the statement is obvious. Suppose that $l \geq 2$. By Lemma 4.4, $\operatorname{supp}_{\mathrm{ess}} \mu_{s_0} = \operatorname{supp}_{\mathrm{ess}} \mu_{t_0}$. Assume that $\operatorname{supp} \mu_{\tilde{s}} = \operatorname{supp} \mu_{\tilde{t}}$ for some $\tilde{s} \neq \tilde{t}$. Then, without loss of generality, we may assume $\operatorname{supp} \mu_{\tilde{s}} = \operatorname{supp} \mu_{\tilde{t}} \neq \operatorname{supp} \mu_{s_0}$. By Lemma 4.4, this implies that $\operatorname{supp}_{\mathrm{ess}} \mu_{\tilde{s}} = \operatorname{supp}_{\mathrm{ess}} \mu_s$ and $\operatorname{supp}_{\mathrm{ess}} \mu_{\tilde{t}} = \operatorname{supp}_{\mathrm{ess}} \mu_s$; that is, $\operatorname{supp}_{\mathrm{ess}} \mu_{\tilde{s}} = \operatorname{supp}_{\mathrm{ess}} \mu_{\tilde{t}}$, which contradicts part (a) of the lemma. Thus, $\operatorname{supp}_{\mathrm{ess}} \mu_1 = ... = \operatorname{supp}_{\mathrm{ess}} \mu_{l+1}$.

Let us prove that the simplex Δ is of the first type, that is, there is a permutation σ such that $\{\sigma\mu_1, ..., \sigma\mu_{l+1}\} = \nabla_1^l$. Set $\mathbf{i} = \operatorname{supp}_{\mathrm{ess}} \mu_1 = ... = \operatorname{supp}_{\mathrm{ess}} \mu_{l+1}$, $\mu_1 = e_{\mathbf{i};j,k,r,s}$. By Lemma 4.4, $\#(\operatorname{supp} \mu_1 \cap \operatorname{supp} \mu_2) = m+2$. There is a unique index $t \in \operatorname{supp} \mu_2$ such that $t \notin \operatorname{supp} \mu_1$. Since $\mu_1|\mu_2$, Lemma 4.4 shows that $\mu_2 \in D = \{e_{\mathbf{i};j,k,r,t}, e_{\mathbf{i};k,j,s,t}, e_{\mathbf{i};r,s,j,t}, e_{\mathbf{i};s,r,k,t}\}$. After a certain Kleinian permutation of the four

last indices j, k, r, s in μ_1, which never changes such a function, and an appropriate renaming the indices in both μ_1 and μ_2, we may assume that $\mu_1 = e_{\mathbf{i};j,k,r,s}$ and $\mu_2 = e_{\mathbf{i};j,k,r,t}$.

Let $p > 2$; let us prove that $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_p = \operatorname{supp}\mu_2 \cap \operatorname{supp}\mu_p$. First, we shall prove that the index t introduced above is not in $\operatorname{supp}\mu_p$. Suppose, on the contrary, that $t \in \operatorname{supp}\mu_p$. Since $\mu_1|\mu_p$, by Lemma 4.4, $\mu_p \in D$. But the set D contains no pair of determinant cross ratios which are proper divisors of each other, which contradicts the condition $\mu_2|\mu_p$. By the same reason, $s \notin \operatorname{supp}\mu_p$. Simple combinatorics show that $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_p = \operatorname{supp}\mu_2 \cap \operatorname{supp}\mu_p$. Furthermore, there is a unique index $t_p \in \operatorname{supp}\mu_p$ such that $t_p \notin \operatorname{supp}\mu_1$. Since $\mu_1|\mu_p$, by Lemma 4.4, $\mu_p \in \{e_{\mathbf{i};j,k,r,t_p}, e_{\mathbf{i};k,j,s,t_p}, e_{\mathbf{i};r,s,j,t_p}, e_{\mathbf{i};s,r,k,t_p}\}$. Since $\operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_2 = \operatorname{supp}\mu_1 \cap \operatorname{supp}\mu_p$, it follows that $\mu_p = e_{\mathbf{i};j,k,r,t_p}$. The group $\mathbf{S}(n)$ is n times transitive on the set $\{1, ..., n\}$, thus, in view of Lemma 2.10, there exists a permutation σ such that $\{\sigma\mu_1, ..., \sigma\mu_{l+1}\} = \nabla_1^l$. This completes the proof of part (b).

$c)$ The action of the permutation group preserves dimension of simplices. By (a) and (b), the $\mathbf{S}(n)$ orbit of any simplex contains a normal simplex. We know that $\dim\nabla_1^t \leq n - m - 3$ and $\dim\nabla_2^t \leq m - 1$ (see Definition 4.5). That is, $\dim L_\vartriangle(\mathcal{E}^n(X)) = \max\{n - (m + 3), m - 1\}$. □

Notice that for $n > m + 3$ the maximal possible dimension of the simplices of the first type is $n - m - 3 \geq 1$; in particular, the normal simplex of the first type ∇_1^{n-m-3} is maximal in the sense of dimension. Similarly, the maximal possible dimension of a simplex of the second type is $m - 1$ and the normal simplex of the second type ∇_2^{m-1} is maximal. In the following two lemmas we describe the stabilizers of the maximal normal simplices ∇_1^{n-m-3} and ∇_2^{m-1} in the group $\mathbf{S}(n)$ acting on the set of all simplices of a given type and dimension; since the latter action is transitive, these lemmas supply us with an important information about the stabilizer of any maximal simplex.

Lemma 4.8. *Let $n > m + 3$. The stabilizer $\operatorname{St}_{\mathbf{S}(n)}(\nabla_1^{n-m-3})$ of the ordered simplex ∇_1^{n-m-3} in the group $\mathbf{S}(n)$ conincides with the subgroup $\mathbf{S}(m-1) = \mathbf{S}(\{1, ..., m-1\}) \subset \mathbf{S}(n)$.*

Proof. For any $s = m + 3, ..., n$, denote $I_s = (1, ..., m + 2, s)$ so that $\nabla_1^{n-m-3} = \{e_{I_s}\}_{s=m+3}^n$. Clearly $\mathbf{S}(m-1) \subset \operatorname{St}_{\mathbf{S}(n)}(\nabla_1^{n-m-3})$. Let $\sigma \in \operatorname{St}_{\mathbf{S}(n)}(\nabla_1^{n-m-3})$. Then

$$\begin{aligned} e_{\{1,...,m-1\};m,m+1,m+2,m+3} &= e_{I_{m+3}} = \sigma e_{I_{m+3}} = \sigma e_{\{1,...,m-1\};m,m+1,m+2,m+3} \\ &= e_{\{\sigma(1),...,\sigma(m-1)\};\sigma(m),\sigma(m+1),\sigma(m+2),\sigma(m+3)} \end{aligned}$$

and hence σ is a disjoint product $\sigma = \phi\psi\theta$, where $\phi \in \mathbf{S}(m-1)$, ψ is one of the four Kleinian permutations Id, $(m, m+1)(m+2, m+3)$, $(m, m+2)(m+1, m+3)$, $(m, m+3)(m+1, m+2)$ and $\theta \in \mathbf{S}(\{m+4, ..., n\})$ (compare to Remark 2.9). For any $t > m + 3$

$$\begin{aligned} e_{\{1,...,m-1\};m,m+1,m+2,t} = e_{I_t} = \sigma e_{I_t} = \phi\psi\theta e_{I_t} &= \phi\psi\theta e_{\{1,...,m-1\};m,m+1,m+2,t} \\ &= e_{\{\phi(1),...,\phi(m-1)\};\psi(m),\psi(m+1),\psi(m+2),\theta(t)} \\ &= e_{\{1,...,m-1\};\psi(m),\psi(m+1),\psi(m+2),\theta(t)}\,. \end{aligned}$$

Consequently, $\psi = \mathrm{Id}$ and $\theta(t) = t$; since the latter is true for any $t > m + 3$, we see that $\theta = \mathrm{Id}$ and $\sigma = \phi \in \mathbf{S}(m-1)$. □

Lemma 4.9. *The stabilizer* $\mathrm{St}_{\mathbf{S}(n)}(\nabla_2^{m-1})$ *of the ordered simplex* ∇_2^{m-1} *in the group* $\mathbf{S}(n)$ *coincides with the subgroup* $\mathbf{S}(\{m+4,...,n\}) \subset \mathbf{S}(n)$.

Proof. Of course, any element of $\mathbf{S}(\{m+4,...,n\})$ does not change ∇_2^{m-1}. Let $\sigma \in \mathrm{St}_{\mathbf{S}(n)}(\nabla_2^{m-1})$. Denote $I = (1,...,m+3)$; then $\nabla_2^{m-1} = \{(i,m)e_I\}_{i=1}^m$, where, as usual, (i,t) denotes the transposition of two indices i,t; furthermore, $(i,m)e_I = \sigma(i,m)e_I$, i.e., $e_I = (i,m)\sigma(i,m)e_I$. For $i = m$ this means that

$$\begin{aligned} e_{\{1,\dots,m-1\};m,m+1,m+2,m+3} &= e_I = \sigma e_I = \sigma e_{\{1,\dots,m-1\};m,m+1,m+2,m+3} \\ &= e_{\{\sigma(1),\dots,\sigma(m-1)\};\sigma(m),\sigma(m+1),\sigma(m+2),\sigma(m+3)} \end{aligned}$$

and hence σ is a disjoint product $\sigma = \theta\phi\psi$, where $\theta \in \mathbf{S}(m-1)$, ϕ is one of the four Kleinian permutations Id, $(m,m+1)(m+2,m+3)$, $(m,m+2)(m+1,m+3)$, $(m,m+3)(m+1,m+2)$ and $\psi \in \mathbf{S}(\{m+4,...,n\})$ (compare to Remark 2.9). For any $i = 1,...,m-1$, we have

$$\begin{aligned} e_{\{1,\dots,i-1,m,i+1,\dots,m-1\};i,m+1,m+2,m+3} &= (i,m)e_{\{1,\dots,m-1\};m,m+1,m+2,m+3} \\ &= (i,m)e_I = \sigma(i,m)e_I = \theta\phi\psi(i,m)e_I \\ &= \theta\phi\psi(i,m)e_{\{1,\dots,m-1\};m,m+1,m+2,m+3} \\ &= e_{\{\theta(1),\dots,\theta(i-1),\phi(m),\theta(i+1),\dots,\theta(m-1)\};\theta(i),\phi(m+1),\phi(m+2),\phi(m+3)}\,. \end{aligned}$$

The latter equality is fulfilled if and only if $\phi = \mathrm{Id}$ and $\theta(i) = i$; since the latter is true for any $i = 1,...,m-1$, we see that $\theta = \mathrm{Id}$ and $\sigma = \psi \in \mathbf{S}(\{m+4,...,n\})$. ☐

4.3. **Maps of $L_\vartriangle(\mathcal{E}^n(X,gp))$ induced by holomorphic endomorphisms.** Here we prove that any strictly equivariant holomorphic map $f\colon \mathcal{E}^n(X,gp) \to \mathcal{E}^n(X,gp)$ induces a simplicial map $f^*\colon L_\vartriangle(\mathcal{E}^n(X,gp)) \to L_\vartriangle(\mathcal{E}^n(X,gp))$. Then we show that, for sufficiently large n, the vertices of certain m simplices of maximal dimension are, up to a permutation of $q_1,..,q_n$, fixed points of the map f^*.

We need the following result similar to Lemma 6.1 and Corollary 6.2 from [11].

Lemma 4.10. *Let X be either $\mathbb{CP}^m$ or $\mathbb{C}^m$. Let $f\colon \mathcal{E}^n(X,gp) \to \mathcal{E}^n(X,gp)$ be a strictly equivariant holomorphic map. If $\lambda \in L(\mathcal{E}^n(X,gp))$ then $\lambda\circ f \in L(\mathcal{E}^n(X,gp))$.*

Proof. It suffices to show that $\lambda \circ f \neq \mathrm{const}$ for any $\lambda \in L(\mathcal{E}^n(X,gp))$. Assume to the contrary that $\lambda \circ f = c \in \mathbb{C} \setminus \{0,1\}$ for some $\lambda \in L_\vartriangle(\mathcal{E}^n(X,gp))$. Then $(\lambda\circ f)(\sigma q) \equiv c$ for all $\sigma \in \mathbf{S}(n)$; as f is strictly equivariant, this implies $\lambda(\sigma f(q)) \equiv c$ for each $\sigma \in \mathbf{S}(n)$. Since $\lambda \in L(\mathcal{E}^n(X,gp))$, it follows that $\lambda^{-1}, 1-\lambda \in L(\mathcal{E}^n(X,gp))$. By Lemma 2.10, there are $s_1, s_2 \in \mathbf{S}(n)$ such that $\lambda^{-1} = s_1\lambda$ and $1-\lambda = s_2\lambda$. Set $\sigma_i = \alpha^{-1}(s_i^{-1})$, where α is an automorphism of $\mathbf{S}(n)$ related to the strictly equivariant mapping f and $i = 1,2$. Clearly, $c^{-1} \equiv \lambda^{-1}(f(q)) \equiv s_1\lambda(f(q)) \equiv \lambda(s_1^{-1}f(q)) \equiv \lambda(f(\sigma_1 q)) \equiv c$ and $1-c \equiv (1-\lambda)(f(q)) \equiv s_2\lambda(f(q)) \equiv \lambda(s_2^{-1}f(q)) \equiv \lambda(f(\sigma_2 q)) \equiv c$. Both of the above equalities can not occur at the same time, this is a contradiction. ☐

Corollary 4.11. *Let $X = \mathbb{CP}^m$ or $\mathbb{C}^m$ and $n > m+2$. Let $f\colon \mathcal{E}^n(X,gp) \to \mathcal{E}^n(X,gp)$ be a strictly equivariant holomorphic map. Then f induces a simplicial mapping f^* whose restriction to each simplex $\Delta \subseteq L(\mathcal{E}^n(X,gp))$ is injective, and dimension of a simplex does not change under this transformation.*

Proof. By Lemma 4.10, For any $\lambda \in L(\mathcal{E}^n(X,gp))$ we have $\lambda \circ f \in L(\mathcal{E}^n(X,gp))$. Thus f induces a map $f^*\colon L(\mathcal{E}^n(X,gp)) \to L(\mathcal{E}^n(X,gp))$ defined by

$$L(\mathcal{E}^n(X,gp)) \ni \lambda \mapsto f^*(\lambda) = \lambda \circ f \in L(\mathcal{E}^n(X,gp))\,.$$

If $\lambda,\mu,\nu\in L(\mathcal{E}^n(X,gp))$ and $\lambda=\mu/\nu$ then

$$f^*(\mu)/f^*(\nu)=f^*(\mu/\nu)=f^*(\lambda)\in L(\mathcal{E}^n(X,gp))\,.$$

Therefore, f^* is a simplicial map, its restriction to each simplex $\Delta\in L_\vartriangle(\mathcal{E}^n(X,gp))$ is injective and dimension of a simplex does not change under this transformation. $\square$

Lemma 4.12. *Let X be either $\mathbb{CP}^m$ or $\mathbb{C}^m$, $n\geq m+3$ and $n\neq 2m+2$. Let $f\colon \mathcal{E}^n(X,gp)\to\mathcal{E}^n(X,gp)$ be a strictly equivariant holomorphic map. Then the induced simplicial map f^* is an automorphism of the complex $L_\vartriangle(\mathcal{E}^n)$. Moreover, f^* preserves the type of simplices.*

Proof. The set $L(\mathcal{E}^n)$ of all vertices of $L_\vartriangle(\mathcal{E}^n)$ is finite. Hence, to prove that the simplicial map f^* is an automorphism of the complex $L_\vartriangle(\mathcal{E}^n)$, it suffices to show that the map $f^*\colon L(\mathcal{E}^n)\to L(\mathcal{E}^n)$ is bijective, which, by the same finiteness reason, is equivalent to its surjectivity.

Let $\alpha\in\operatorname{Aut}(\mathbf{S}(n))$ be the automorphism related to our strictly equivariant holomorphic map f so that $f(\sigma q)=\alpha(\sigma)f(q)$ and $f^*(\sigma\mu)=\alpha^{-1}(\sigma)[f^*(\mu)]$ for all $\sigma\in\mathbf{S}(n)$ and $\mu\in L(\mathcal{E}^n)$.

Let $\mu\in L(\mathcal{E}^n)$ and $\nu=f^*(\mu)$. By Lemma 2.10, there exists $\sigma\in\mathbf{S}(n)$ such that $\sigma\nu=\mu$. Set $\lambda=\alpha(\sigma)\mu\in L(\mathcal{E}^n)$; then $\mu=\sigma\nu=\sigma(f^*(\mu))=f^*(\alpha(\sigma)\mu)=f^*(\lambda)$, which proves that the map $f^*\colon L(\mathcal{E}^n)\to L(\mathcal{E}^n)$ is surjective and thereby bijective. It follows that f^* is a simplicial automorphism of the finite complex $L_\vartriangle(\mathcal{E}^n)$.

Let us prove now that f^* preserves the type of simplices. First assume that $n>2m+2$. We start with the normal simplex ∇_1^{n-m-3} and its faces. Since f^* preserves dimension, $\dim f^*(\nabla_1^{n-m-3})=n-m-3>m-1$ and Lemma 4.7(a,b) shows that the simplex $f^*(\nabla_1^{n-m-3})$ is of the first type. Any normal simplex of the first type ∇_1^l is a face of ∇_1^{n-m-3} and any face of a simplex of the first type is also a simplex of the first type (see Definition 4.5). Since $f^*(\nabla_1^l)$ is a face of the simplex $f^*(\nabla_1^{n-m-3})$ which is of the first type, $f^*(\nabla_1^l)$ is of the first type.

Now let $\Delta\in L_\vartriangle(\mathcal{E}^n)$ be any l-simplex of the first type. It follows from Definition 4.5 that there is $\sigma\in\mathbf{S}(n)$ such that $\sigma\nabla_1^l=\Delta$. Therefore the simplex $f^*(\Delta)=f^*(\sigma\nabla_1^l)=\alpha^{-1}(\sigma)f^*(\nabla_1^l)$ is of the first type. Thus, f^* carries simplices of the first type to simplices of the first type.

Since the simplicial map f^* is an automorphism of the finite complex $L_\vartriangle(\mathcal{E}^n)$, f^* is bijective on the set of all simplices of positive dimension. By Lemma 4.7(a,b) and Definition 4.5, the latter set is a disjoint union of two its subsets consisting of all simplices of the first and the second type, respectively. Therefore, it follows from what was proved above that f^* carries simplices of the second type to simplices of the second type. This completes the proof of the case $n>2m+2$

When $n<2m+2$, we consider the normal simplex ∇_2^{m-1} and its faces. Now we have $\dim f^*(\nabla_2^{m-1})=m-1>n-m-3$ and, by Lemma 4.7(a,b), the simplex $f^*(\nabla_2^{m-1})$ is of the second type. Any normal simplex of the second type ∇_2^l is a face of ∇_2^{m-1} and any face of a simplex of the second type is also a simplex of the second type (see Definition 4.5). Since $f^*(\nabla_2^l)$ is a face of the simplex $f^*(\nabla_2^{m-1})$ of the second type, $f^*(\nabla_2^l)$ is also of the second type. Thus, f^* preserves the type of the simplices of the second type.

Any l-simplex $\Delta\in L_\vartriangle(\mathcal{E}^n)$ of the second type may be represented as $\Delta=\sigma\nabla_2^l$ with some $\sigma\in\mathbf{S}(n)$. Therefore the simplex $f^*(\Delta)=f^*(\sigma\nabla_2^l)=\alpha^{-1}(\sigma)f^*(\nabla_2^l)$

is of the second type. Precisely as above, using the bijectivity of f^* on the set of all simplices of positive dimension and taking into account Lemma 4.7(a, b) and Definition 4.5, we come to conclusion that f^* carries simplices of the first type to simplices of the first type, which completes the proof of the lemma. □

REMARK 4.13. Let $\alpha \in \operatorname{Aut} \mathbf{S}(n)$ be the automorphism related to a strictly equivariant holomorphic endomorphism f of $\mathcal{E}^n$. It is easily seen that for any $\rho \in \mathbf{S}(n)$ and any function λ on $\mathcal{E}^n$ we have

$$(\alpha(\rho)f)^*(\lambda) = \lambda \circ [\alpha(\rho)f)] = \lambda \circ f \circ \rho = \rho^{-1}[\lambda \circ f] = \rho^{-1}[f^*(\lambda)]\,, \tag{4.3}$$

where $\rho^{-1}[\lambda \circ f]$ is the result of the action of the permutation ρ^{-1} on the function $\lambda \circ f = f^*(\lambda)$. Changing ρ with ρ^{-1}, we obtain

$$\rho[f^*(\lambda)] = (\alpha(\rho^{-1})f)^*(\lambda) = \lambda \circ [\alpha(\rho^{-1})f)] = (\alpha(\rho)\lambda) \circ f = f^*(\alpha(\rho)\lambda)\,. \tag{4.4}$$

○

REMARK 4.14. Let $f = f_{\tau,\sigma}\colon \mathcal{E}^n \to \mathcal{E}^n$ be a strictly equivariant tame holomorphic map and let $\mu \in L(\mathcal{E}^n)$. Since the function μ is $\mathbf{PSL}(m+1, \mathbb{C})$ invariant, we have $(f^*\mu)(q) = \mu(f(q)) = \mu(\sigma\tau(q)q) = \mu(\tau(q)\sigma q) = \mu(\sigma q)$. Taking into account that f is strictly equivariant, we obtain

$$\mu(q) = (f^*\mu)(\sigma^{-1}q) = \mu(f(\sigma^{-1}q)) = \mu(\alpha(\sigma^{-1})f(q)) = [(\alpha(\sigma^{-1})f)^*\mu](q)\,. \tag{4.5}$$

The latter formula shows that all the vertices $\mu \in L(\mathcal{E}^n)$ of the complex $L_\vartriangle(\mathcal{E}^n)$ are fixed points of the map $(\alpha(\sigma^{-1})f)^*$; in other words, *the action of the mapping f^* itself on the set $L(\mathcal{E}^n)$ of all vertices of the complex $L_\vartriangle(\mathcal{E}^n)$ coincides with the action of a certain permutation.*

In the theorem below we prove a weaker property of all strictly equivariant holomorphic endomorphisms f of $\mathcal{E}^n$. It says that for any such f there is a permutation $\rho \in \mathbf{S}(n)$ such that the vertices of certain special simplices of $L_\vartriangle(\mathcal{E}^n)$ are fixed points of the map $(\rho f)^*$; that is, the action of f^* on those particular vertices of $L_\vartriangle(\mathcal{E}^n)$ coincides with the action of a certain permutation. This prelimenary result will play important part in the proof of Theorem 1.4. ○

According to Notation 2.13, $\mathbf{m}(\hat{s}) = (1, ..., \hat{s}, ..., m)$ for any $s = 1, .., m$; sometimes we write $\widehat{\mathbf{m}}$ instead of $\mathbf{m}(\hat{m})$. We use the standard notation (i, j) for the transposition of two distinct elements $i, j \in \{1, ..., n\}$. Notice that for $i = 1, ..., m$ the permutation $(i, m)\nabla_1^{n-m-3}$ of the simplex $\nabla_1^{n-m-3} = \{e_{\widehat{\mathbf{m}};m,m+1,m+2,s}\}_{s=m+3}^n$ is the simplex $\{e_{\mathbf{m}(\hat{\imath});i,m+1,m+2,s}\}_{s=m+3}^n$.

THEOREM 4.15. *Let X be either $\mathbb{CP}^m$ or $\mathbb{C}^m$, $n \geq m+3$ and $n \neq 2m+2$. Let $f\colon \mathcal{E}^n(X, gp) \to \mathcal{E}^n(X, gp)$ be a strictly equivariant holomorphic map. There exists a permutation $\rho \in \mathbf{S}(n)$ such that $(\rho f)^*(e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}) = e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}$ for any $r \in \{1, ..., m\}$ and $s \in \{m+3, ..., n\}$. In other words, the map $(\rho f)^*$ is identical on each of the simplices*

$$\{e_{\mathbf{m}(\hat{1});1,m+1,m+2,s}\}_{s=m+3}^n, ..., \{e_{\mathbf{m}(\hat{m});m,m+1,m+2,s}\}_{s=m+3}^n\,. \tag{4.6}$$

Proof. First assume that $n > m+3$. Notice that the last simplex in the above list, namely

$$\{e_{\mathbf{m}(\hat{m});m,m+1,m+2,s}\}_{s=m+3}^n = \{e_{\widehat{\mathbf{m}};m,m+1,m+2,s}\}_{s=m+3}^n = \nabla_1^{n-m-3}\,,$$

is the normal $(n-m-3)$-simplex of the first type (see Definition 4.5). To simplify the notation, for any $s = m+3, ..., n$, let us set $I_s = (1, ..., m+2, s)$; notice that I_s

is the support of the determinant cross ratio $e_{\widehat{\mathbf{m}};m,m+1,m+2,s}$, which is one of the vertices of ∇_1^{n-m-3}.

By Lemma 4.12, f^* preserves the type of simplices; hence $f^*(\nabla_1^{n-m-3})$ is a simplex of the first type and there is a permutation θ such that $(\theta f)^*(\nabla_1^{n-m-3}) = \nabla_1^{n-m-3}$. Clearly, it would be sufficient to prove the theorem for the strictly equivariant map θf. Therefore, without loss of generality, we may from the very beginning assume that our original map f satisfies $f^*(\nabla_1^{n-m-3}) = \nabla_1^{n-m-3}$. Since we deal with ordered simplices, the latter relation means that all vertices of the simplex ∇_1^{n-m-3} are fixed points of the map f^*, that is,

$$f^*(e_{\widehat{\mathbf{m}};m,m+1,m+2,s}) = e_{\widehat{\mathbf{m}};m,m+1,m+2,s}\,,$$

or, which is the same,

$$f^*(e_{I_s}) = e_{I_s} \quad \text{for all} \quad s = m+3, ..., n\,, \tag{4.7}$$

Since f is strictly equivariant, there is $\alpha \in \mathrm{Aut}(\mathbf{S}(n))$ such that $f(\sigma q) = \alpha(\sigma) f(q)$ for every $\sigma \in \mathbf{S}(n)$. Consequently, for $1 \leq i < m$ and $m \leq t \leq n$ we have

$$f^*((i,t)\nabla_1^{n-m-3}) = \alpha^{-1}((i,t))\nabla_1^{n-m-3}\,, \tag{4.8}$$

where (i,t) is the transposition of i and t.

For any $s = m+3, ..., n$, set $I_s = (1, ..., m+2, s)$; notice that I_s is the support of the determinant cross ratio $e_{\widehat{\mathbf{m}};m,m+1,m+2,s}$, which is one of the vertices of ∇_1^{n-m-3}.

The permuted simplices

$$\begin{aligned}
\Delta_{s,1} &= (m+3,s)\nabla_2^{m-1} = \{(1,m)e_{I_s}, (2,m)e_{I_s}, ..., (m-1,m)e_{I_s}, e_{I_s}\}\,,\\
\Delta_{s,2} &= (m+3,s)(m,m+1)(m+2,m+3)\nabla_2^{m-1}\\
&= \{(1,m+1)e_{I_s}, (2,m+1)e_{I_s}, ..., (m-1,m+1)e_{I_s}, e_{I_s}\}\,,\\
\Delta_{s,3} &= (m+3,s)(m,m+2)(m+1,m+3)\nabla_2^{m-1}\\
&= \{(1,m+2)e_{I_s}, (2,m+2)e_{I_s}, ..., (m-1,m+2)e_{I_s}, e_{I_s}\}\,,\\
\Delta_{s,4} &= (m+3,s)(m,m+3)(m+1,m+2)\nabla_2^{m-1}\\
&= \{(1,s)e_{I_s}, (2,s)e_{I_s}, ..., (m-1,s)e_{I_s}, e_{I_s}\}
\end{aligned}$$

are the simplices of the second type. By Lemma 4.12, all $f^*(\Delta_{s,\kappa})$, $\kappa = 1,2,3,4$, are simplices of the second type. Since f^* preserves vertices of ∇_1^{n-m-3}, we have $f^*(e_{I_s}) = e_{I_s}$ and hence each of the simplices $f^*(\Delta_{s,\kappa})$, $\kappa = 1,2,3,4$, contains the vertex e_{I_s} whose essential support is $\widehat{\mathbf{m}} = (1, ..., m-1)$. By Lemma 4.7(a), the essential supports of all vertices of $f^*(\Delta_{s,\kappa})$ but e_{I_s} are not equal to $\mathrm{supp}_{\mathrm{ess}}\, e_{I_s} = \widehat{\mathbf{m}}$.

Claim. *There exists $\sigma \in \mathbf{S}(m-1) = \mathbf{S}(\{1, ..., m-1\}) \subset \mathbf{S}(n)$ such that for any $s > m+2$ the couple of the simplices $f^*(\Delta_{s,2})$, $f^*(\Delta_{s,3})$ coincides with the couple of the simplices $\sigma\Delta_{s,2}$, $\sigma\Delta_{s,3}$.*

Proof of Claim. We divide the proof to four steps.

Step 1. Pick some $s \geq m+3$. Let

$$\begin{aligned}
\Delta_s &= f^*(\Delta_{s,2})\\
&= \{f^*((1,m+1)e_{I_s}), f^*((2,m+1)e_{I_s}), ..., f^*((m-1,m+1)e_{I_s}), f^*(e_{I_s})\}\\
&= \{f^*((1,m+1)e_{I_s}), f^*((2,m+1)e_{I_s}), ..., f^*((m-1,m+1)e_{I_s}), e_{I_s}\}\,;
\end{aligned}$$

then Δ_s is a simplex of the second type and Lemma 4.7(a,b) shows that

$$\operatorname{supp} f^*((1,m+1)e_{I_s}) = ... = \operatorname{supp} f^*((m-1,m+1)e_{I_s}) = \operatorname{supp} e_{I_s} = I_s\,. \quad (4.9)$$

The simplex $\Delta_{s,2}$ is also of the second type and, according to Definition 4.5, it may be carried to the simplex Δ_s by a permutation $\phi_s \in \mathbf{S}(n)$. The vertices of both these simplices depend only on the vector variables $q_1, ..., q_{m+2}, q_s$ (see Section 2.1 and Definition 2.8); by Remark 4.6, ϕ_s may be chosen in the subgroup $\mathbf{S}(I_s) = \mathbf{S}(\{1, ..., m+2, s\}) \subset \mathbf{S}(n)$ and, by Lemma 4.9, such a permutation is unique. In particular, we have

$$\begin{aligned} e_{I_s} &= \phi_s e_{I_s} = \phi_s e_{\{1,...,m-1\};m,m+1,m+2,s} \\ &= e_{\{\phi_s(1),...,\phi_s(m-1)\};\phi_s(m),\phi_s(m+1),\phi_s(m+2),\phi_s(s)} \end{aligned}$$

and hence ϕ_s is a disjoint product $\phi_s = \sigma_s\theta_s$, where $\sigma_s \in \mathbf{S}(m-1)$ and θ_s is one of the four Kleinian permutations Id, $(m,m+1)(m+2,s)$, $(m,m+2)(m+1,s)$, $(m,s)(m+1,m+2)$ (compare to Remark 2.9). Consequently, for any $i = 1, ..., m-1$ we have

$$\begin{aligned} \theta_s(i,m+1)e_{I_s} &= \theta_s(i,m+1)e_{\{1,...,m-1\};m,m+1,m+2,s} \\ &= \theta_s e_{\{1,...,i-1,m+1,i+1,...,m-1\};m,i,m+2,s} \\ &= e_{\{1,...,i-1,\theta_s(m+1),i+1,...,m-1\};\theta_s(m),i,\theta_s(m+2),\theta_s(s)}\,; \end{aligned} \quad (4.10)$$

the latter function, in turn, must be one of the determinant cross ratios

$$(i,m+1)e_{I_s}\,, \quad (i,m)e_{I_s}\,, \quad (i,s)e_{I_s}\,, \quad (i,m+2)e_{I_s}\,.$$

This means that $\theta_s\Delta_{s,2} = \Delta_{s,j_s}$ for a certain $j_s \in \{1,2,3,4\}$ and hence

$$\Delta_s = f^*(\Delta_{s,2}) = \sigma_s\theta_s\Delta_{s,2} = \sigma_s\Delta_{s,j_s}\,. \quad (4.11)$$

Since $\sigma_s \in \mathbf{S}(m-1)$ does not touch the indices $m, m+1, ..., n$, relation (4.11) determines both j_s and θ_s; moreover, j_s is uniquely determined by the value of θ_s on any one of the numbers $m, m+1, m+2, s$. In particular, we see that the value $\theta_s(m+1) \in \{m, m+1, m+2, s\}$ determines j_s; more precisely,

(i) the ordered couple $(j_s; \theta_s(m+1))$ is one of the four ordered couples

$$(1;m)\,,\ (2;m+1)\,,\ (3;m+2)\,,\ (4,s)\,. \quad (4.12)$$

Step 2. Let us show now that both the permutation $\sigma_s \in \mathbf{S}(m-1)$ and the index $j_s \in \{1,2,3,4\}$ in (4.11) do not depend on s and, moreover, $j_s \neq 4$.

We start with σ_s. Fix some $i \in \{1, ..., m-1\}$. According to (4.8), the simplex $f^*((i,m+1)\nabla_1^{n-m-3}) = \alpha^{-1}((i,m+1))\nabla_1^{n-m-3}$ is of the first type; the functions

$$f^*((i,m+1)e_{I_{m+3}}), f^*((i,m+1)e_{I_{m+4}}), ..., f^*((i,m+1)e_{I_n})$$

are its vertices, and Lemma 4.7(a,b) shows that

$$\operatorname{supp}_{ess} f^*((i,m+1)e_{I_{m+3}}) = ... = \operatorname{supp}_{ess} f^*((i,m+1)e_{I_n})\,. \quad (4.13)$$

Furthermore, the function $f^*((i,m+1)e_{I_s})$ is the i^{th} vertex of the simplex $f^*(\Delta_{s,2})$; by (4.11), $f^*(\Delta_{s,2}) = \sigma_s\theta_s\Delta_{s,2}$ and hence, by (4.10),

$$\begin{aligned} f^*((i,m+1)e_{I_s}) &= \sigma_s\theta_s(i,m+1)e_{I_s} \\ &= \sigma_s e_{\{1,...,i-1,\theta_s(m+1),i+1,...,m-1\};\theta_s(m),i,\theta_s(m+2),\theta_s(s)} \\ &= e_{\{\sigma_s(1),...,\sigma_s(i-1),\theta_s(m+1),\sigma_s(i+1),...,\sigma_s(m-1)\};\theta_s(m),\sigma_s(i),\theta_s(m+2),\theta_s(s)}\,. \end{aligned}$$

Therefore, by (4.13), the set

$$\begin{aligned}\Sigma &\overset{\text{def}}{=\!=} \operatorname{supp}_{ess} f^*((i,m+1)e_{I_s})\\ &= \{\sigma_s(1),...,\sigma_s(i-1),\theta_s(m+1),\sigma_s(i+1),...,\sigma_s(m-1)\}\end{aligned} \tag{4.14}$$

does not depend on s. The only element of Σ that is not in $\{1,...,m-1\}$ is $\theta_s(m+1)$; hence $\theta_s(m+1)$ does not depend on s and the same is true for the set $\Sigma' \overset{\text{def}}{=\!=} \{\sigma_s(1),...,\widehat{\sigma_s(i)},...,\sigma_s(m-1)\}$ of all elements of Σ but $\theta_s(m+1)$. In fact the set Σ' consists of all numbers $1,...,m-1$ but $\sigma_s(i)$; thus, $\sigma_s(i)$ also does not depend on s. Since this is the case for any $i \in \{1,...,m-1\}$, the the permutation $\sigma \overset{\text{def}}{=\!=} \sigma_s \in \mathbf{S}(m-1)$ does not depend on s.

Now we turn to the index $j_s \in \{1,2,3,4\}$. θ_s is a Kleinian permutation of $m, m+1, m+2, s$ and we have already proved that the element

$$\theta_s(m+1) \in \{m, m+1, m+2, s\}$$

does not depend on $s \in \{m+3,...,n\}$; thus,

$$\theta_{m+3}(m+1) = ... = \theta_n(m+1) \in \bigcap_{s=m+3}^{n} \{m, m+1, m+2, s\} = \{m, m+1, m+2\}$$

and hence $\theta_s(m+1) \neq s$. According to (i), this means that $j_s \neq 4$ and j_s does not depend on s.

Thus, as a result of steps 1 and 2, we know that

(ii) there are a permutation $\sigma \in \mathbf{S}(m-1)$ and $j \in \{1,2,3\}$ such that

$$f^*(\Delta_{s,2}) = \sigma\Delta_{s,j} \quad \text{for any} \quad s \in \{m+3,...,n\}\,. \tag{4.15}$$

Step 3. In a similar way, one can show that

(ii') there are a permutation $\sigma' \in \mathbf{S}(m-1)$ and $j' \in \{1,2,3\}$ such that

$$f^*(\Delta_{s,3}) = \sigma'\Delta_{s,j'} \quad \text{for any} \quad s \in \{m+3,...,n\}\,. \tag{4.16}$$

Step 4. Let us prove now that $\sigma = \sigma'$, $j, j' \neq 1$ and $j \neq j'$; this will complete the proof of Claim.

By Lemma 3.6(d), for any $s \in \{m+3,...n\}$ the function e_{I_s} admits the following two representations as products of two determinant cross ratios:

$$e_{I_s} = ((i,m)e_{I_s}) \cdot ((i,s)e_{I_s}) = ((i,m+1)e_{I_s}) \cdot ((i,m+2)e_{I_s})\,. \tag{4.17}$$

Hence, by (4.7)

$$\begin{aligned}e_{I_s} &= f^*(e_{I_s}) = f^*((i,m)e_{I_s}) f^*((i,s)e_{I_s})\\ &= f^*((i,m+1)e_{I_s}) f^*((i,m+2)e_{I_s})\,.\end{aligned} \tag{4.18}$$

Suppose that $j = 1$. Then, according to (4.15), $f^*((i,m+1)e_{I_s})$ is the i^{th} vertex of the simplex $f^*(\Delta_{s,2}) = \sigma\Delta_{s,1}$ and hence $f^*((i,m+1)e_{I_s}) = (\sigma(i),m)e_{I_s}$. By (4.18) and (4.17), we have

$$f^*((i,m+2)e_{I_s}) = \frac{e_{I_s}}{f^*((i,m+1)e_{I_s})} = \frac{e_{I_s}}{(\sigma(i),m)e_{I_s}} = (\sigma(i),s)e_{I_s}\,. \tag{4.19}$$

By (4.16) and (4.19), the function $(\sigma(i),s)e_{I_s}$ must be the i^{th} vertex of the simplex $f^*(\Delta_{s,3}) = \sigma'\Delta_{s,j'}$; the latter shows that $j' = 4$, which contradicts (ii'). Thus $j \neq 1$.

The proof of the inequality $j' \neq 1$ is similar to the above one. Consequently, $j, j' \in \{2, 3\}$.

Now we turn to the permutations σ and σ'.

Suppose first that $j = 2$. Then, according to (4.15), each $f^*((i, m+1)e_{I_s})$, $i = 1, ..., m-1$, is a vertex of the simplex $f^*(\Delta_{s,2}) = \sigma\Delta_{s,2}$; hence $f^*((i, m+1)e_{I_s}) = (\sigma(i), m+1)e_{I_s}$. By (4.18) and (4.17), we have

$$f^*((i, m+2)e_{I_s}) = \frac{e_{I_s}}{f^*((i, m+1)e_{I_s})} = \frac{e_{I_s}}{(\sigma(i), m+1)e_{I_s}} = (\sigma(i), m+2)e_{I_s}\,. \quad (4.20)$$

By (4.16) and (4.20), the function $(\sigma(i), m+2)e_{I_s}$ must be the i^{th} vertex of the simplex $f^*(\Delta_{s,3}) = \sigma'\Delta_{s,j'}$. The latter shows that $j' = 3 \neq j$ and $\sigma'(i) = \sigma(i)$ for any i and hence $\sigma' = \sigma$.

Finally, for $j = 3$, in the same way as above, we obtain $j' = 2 \neq j$ and $\sigma' = \sigma$. This completes Step 4 and proves Claim.

Continuing the proof of the theorem, notice that by almost the same argument as in the Steps 1 and 2 above, one can show that there is a permutation $\vartheta \in \mathbf{S}(m-1) \subset \mathbf{S}(n)$ and an index $l \in \{1, 2, 3\}$ such that

$$f^*(\Delta_{s,1}) = \vartheta\Delta_{s,l} \quad \text{for any} \quad s \geq m+3\,. \quad (4.21)$$

It follows from Claim that either $l = 1$ or $l \in \{2, 3\} = \{j, j'\}$, where j and j' are defined by (ii) and (ii') and, according to Step 4 of the proof above, are distinct elements of the set $\{2, 3\}$.

Thus, we must consider the following three cases: $a)$ $l = 1$, $b)$ $l = j$ and $c)$ $l = j'$.

$a)$ In this case (4.21) takes the form $f^*(\Delta_{s,1}) = \vartheta\Delta_{s,1}$; therefore, using (4.3) for all vertices μ of the simplex $\Delta_{s,1}$ and the permutation $\rho = \vartheta$, we obtain

$$(\alpha(\vartheta)f)^*(\Delta_{s,1}) = \vartheta^{-1}[f^*(\Delta_{s,1})] = \vartheta^{-1}\vartheta\Delta_{s,1} = \Delta_{s,1}\,; \quad (4.22)$$

in terms of the vertices of the ordered simplices, this means that for any $i = 1, ..., m-1$

$$(\alpha(\vartheta)f)^*((i, m)e_{I_s}) = (i, m)e_{I_s}$$

and

$$(\alpha(\vartheta)f)^*(e_{I_s}) = e_{I_s}$$

for all $s \geq m+3$. Since $e_{I_{m+3}}, ..., e_{I_n}$ are all the vertices of the simplex ∇_1^{n-m-3}, we see that

$$(\alpha(\vartheta)f)^*(\nabla_1^{n-m-3}) = \nabla_1^{n-m-3}$$

and

$$(\alpha(\vartheta)f)^*((i, m)\nabla_1^{n-m-3}) = (i, m)\nabla_1^{n-m-3}$$

for all $i = 1, ..., m-1$; this proves the theorem in the case $a)$.

Let us prove now that the cases $b)$ and $c)$ are impossible. Indeed, if $l = j$ then (4.21) and (ii) imply that $f^*(\Delta_{s,1}) = \varphi[f^*(\Delta_{s,2})]$ with the permutation $\varphi = \vartheta\sigma^{-1} \in \mathbf{S}(m-1) \subset \mathbf{S}(n)$ that does not depend on s. Therefore, using (4.4) for the vertices μ of the simplex $\Delta_{s,2}$ and the permutation $\rho = \varphi$, we obtain $f^*(\Delta_{s,1}) = f^*(\alpha(\varphi)\Delta_{s,2})$; since f^* is an automorphism of the complex $L_\vartriangle(\mathcal{E}^n)$ (see Lemma 4.12), the latter relation implies

$$\Delta_{s,1} = \alpha(\varphi)\Delta_{s,2}\,. \quad (4.23)$$

Notice that e_{I_s} is the very last vertex of the $(m-1)$-simplices $\Delta_{s,1}$ and $\Delta_{s,2}$; hence for all $s \geq m+3$ we have $e_{I_s} = \alpha(\varphi)e_{I_s}$. Since $e_{I_{m+3}}, ..., e_{I_n}$ are all the vertices of the simplex ∇_1^{n-m-3}, we see that $\nabla_1^{n-m-3} = \alpha(\varphi)\nabla_1^{n-m-3}$. Lemma 4.8 implies

that $\alpha(\varphi) \in \mathbf{S}(m-1) \subset \mathbf{S}(n)$. As $m > 1$, one can see from the definition of $\Delta_{s,1}$ and $\Delta_{s,2}$ that $\Delta_{s,1} \neq \psi\Delta_{s,2}$ for any $\psi \in \mathbf{S}(m-1)$, which contradicts to (4.23). The case c) may be treated similarly. This completes the proof of the theorem in the case $n > m+3$.

When $n = m+3$, the theorem asserts that there is a permutation $\rho \in \mathbf{S}(n)$ such that $(\rho f)^*(e_{\mathbf{m}(\hat{r});r,m+1,m+2,m+3}) = e_{\mathbf{m}(\hat{r});r,m+1,m+2,m+3}$ for any $r \in \{1,...,m\}$. The functions $e_{\mathbf{m}(\hat{1});1,m+1,m+2,m+3}$,..., $e_{\mathbf{m}(\hat{m});m,m+1,m+2,m+3}$ are all the vertices of the ordered simplex ∇_2^{m-1}. Thus, the statement in question says that $(\rho f)^*(\nabla_2^{m-1}) = \nabla_2^{m-1}$ for an appropriate permutation $\rho \in \mathbf{S}(n)$. Since f is strictly equivariant, this is equivalent to the existence of $\phi \in \mathbf{S}(n)$ such that $f^*(\nabla_2^{m-1}) = \phi\nabla_2^{m-1}$, which means precisely that the simplex $f^*(\nabla_2^{m-1})$ is of the second type. The latter property follows from Lemma 4.12, which completes the proof. $\square$

5. The proof of Theorem 1.4

Here we prove the main result of this paper. We start with the following remark which is similar to Remark 2.14 in [11].

Remark 5.1. For $n \geq m+3$, there exists a non-empty Zariski open subset $U \subset \mathcal{E}^n(X, gp)$ such that if $Aq = \sigma q$ for some $q \in U$, $A \in \mathbf{PSL}(m+1, \mathbb{C})$ and $\sigma \in \mathbf{S}(n)$ then $A = \mathrm{Id}$ and $\sigma = \mathrm{Id}$.

Indeed, Lemma 2.5 implies that for any two points $q = (q_1, ..., q_n)$ and $q' = (q'_1, ..., q'_n)$ in $\mathcal{E}^n(X, gp)$, an element $A \in \mathbf{PSL}(m+1, \mathbb{C})$ is uniquely determined by the requirement $Aq_i = q'_i$ for all $i = 1, ..., m+2$. Since $\mathbf{S}(n)$ is finite, it follows that the set S of all points $q = (q_1, ..., q_n) \in \mathcal{E}^n(X, gp)$ such that for some $A \in \mathbf{PSL}(m+1, \mathbb{C})$ and some non-trivial permutation $\sigma \in \mathbf{S}(n)$ (both A and σ may depend on q) the point $Aq = (Aq_1, ..., Aq_n)$ coincides with the permuted point $\sigma q = (q_{\sigma^{-1}(1)}, ..., q_{\sigma^{-1}(n)})$ is a proper Zariski closed subset of $\mathcal{E}^n(X, gp)$. Its complement $U = \mathcal{E}^n(X, gp) \setminus S$ is a desired non-empty Zariski open set. $\bigcirc$

5.1. **Proof of Theorem 1.4.** According to Theorem 4.15, there exists a permutation ρ such that

$$e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(\rho f(q)) = e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(q) \tag{5.1}$$

for all $q \in \mathcal{E}^n(X, gp)$, $s = m+3, ..., n$ and $r = 1, ..., m$. Lemma 2.5 implies that there exists a map $\gamma\colon \mathcal{E}^n(\mathbb{CP}^m, gp) \to \mathbf{PSL}(m+1, \mathbb{C})$ such that $\gamma(q)q \in M_{m,n}$ (see Definition 2.4). Lemma 2.11 says that determinant cross ratios are $\mathbf{PSL}(m+1, \mathbb{C})$-invariant; therefore

$$e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(\gamma(q)q) = e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(q) \tag{5.2}$$

and

$$e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(\gamma(\rho f(q))\rho f(q)) = e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(\rho f(q)) \tag{5.3}$$

for all $q \in \mathcal{E}^n(X, gp)$, $s \in \{m+3, \ldots, n\}$ and $r \in \{1, \ldots, m\}$. Comparing (5.1), (5.2) and (5.3) we obtain that

$$e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(\gamma(q)q) = e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}(\gamma(\rho f(q))\rho f(q)) \tag{5.4}$$

for any $q \in \mathcal{E}^n(X, gp)$ and all $s \in \{m+3, \ldots, n\}$ and $r \in \{1, \ldots, m\}$. Both points $\gamma(q)q$ and $\gamma(\rho f(q))(\rho f(q))$ are in $M_{m,n}$, and Lemma 2.14 says that the functions $e_{\mathbf{m}(\hat{r});r,m+1,m+2,s}$ with $s \in \{m+3, \ldots, n\}$ and $r \in \{1, \ldots, m\}$ separate points of

$M_{m,n}$. Consequently, (5.4) implies that $\gamma(\rho f(q))\rho f(q) = \gamma(q)q$, or, which is the same, $\rho f(q) = (\gamma(\rho f(q)))^{-1}\gamma(q)q$. Set $\tau(q) = (\gamma(\rho f(q)))^{-1}\gamma(q)$; the map

$$\tau\colon \mathcal{E}^n(X, gp) \ni q \mapsto \tau(q) \in \mathbf{PSL}(m+1, \mathbb{C})$$

is holomorphic and $\tau(q)q = \rho f(q)$, that is, $f(q) = \sigma\tau(q)q$ for all $q \in \mathcal{E}^n(X, gp)$, where $\sigma = \rho^{-1} \in \mathbf{S}(n)$.

To complete the proof, we must check that the morphism τ is $\mathbf{S}(n)$-invariant. Let $\alpha \in \operatorname{Aut}\mathbf{S}(n)$ be the automorphism related to our strictly equivariant map f. For every $\theta \in \mathbf{S}(n)$ and all $q \in \mathcal{E}^n(X, gp)$ we have

$$\sigma\tau(\theta q)\theta q = f(\theta q) = \alpha(\theta)f(q) = \alpha(\theta)\sigma\tau(q)q\,,$$

which can be written as

$$[(\tau(\theta q))^{-1} \cdot \tau(q)]q = \sigma^{-1}\alpha(\theta^{-1})\sigma\theta q\,, \tag{5.5}$$

where $(\tau(\theta q))^{-1} \cdot \tau(q) \in \mathbf{PSL}(m+1, \mathbb{C})$ means the product of two elements of the group $\mathbf{PSL}(m+1, \mathbb{C})$. In view of Remark 5.1, this implies that

$$\sigma^{-1}\alpha(\theta^{-1})\sigma\theta = \mathrm{Id} \quad \text{and} \quad \tau(\theta q) = \tau(q) \tag{5.6}$$

for all $\theta \in \mathbf{S}(n)$ and all q in a non-empty Zariski open subset $U \subset \mathcal{E}^n(X, gp)$. Since τ is continuous, the latter relation holds true for all $q \in \mathcal{E}^n(X, gp)$; since $\theta \in \mathbf{S}(n)$ was arbitrary, this shows that the morphism $\tau\colon \mathcal{E}^n(X, gp) \to \mathbf{PSL}(m+1, \mathbb{C})$ is $\mathbf{S}(n)$-invariant. This completes the proof of Theorem 1.4. ☐

The following statement is an obvious corollary of Theorem 1.4 and Definition 1.1.

Corollary 5.2. *Let $m > 1$, $n \geq m+3$ and $n \neq 2m+2$.*

a) Any holomorphic map $F\colon \mathcal{C}^n(\mathbb{CP}^m, gp) \to \mathcal{C}^n(\mathbb{CP}^m, gp)$ that can be lifted to a strictly equivariant holomorphic map $f\colon \mathcal{E}^n(\mathbb{CP}^m, gp) \to \mathcal{E}^n(\mathbb{CP}^m, gp)$ is tame.

b) Any holomorphic map $F\colon \mathcal{C}^n(\mathbb{C}^m, gp) \to \mathcal{C}^n(\mathbb{C}^m, gp)$ that can be lifted to a strictly equivariant holomorphic map $f\colon \mathcal{E}^n(\mathbb{C}^m, gp) \to \mathcal{E}^n(\mathbb{C}^m, gp)$ is quasitame.

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